PCF denotational semantics — aims

- PCF types $\tau \;\mapsto\;$ domains $[\![\tau]\!]$.

- Closed PCF terms $M : \tau \;\mapsto\;$ elements $[\![M]\!] \in [\![\tau]\!]$.

  Denotations of open terms will be continuous functions.

- Compositionality.

  In particular: $[\![M]\!] = [\![M']\!] \;\Rightarrow\; [\![\mathcal{C}[M]]\!] = [\![\mathcal{C}[M']]\!]$.

- Soundness.

  For any type τ, $M \Downarrow_\tau V \;\Rightarrow\; [\![M]\!] = [\![V]\!]$.

- Adequacy.

  For $\tau = bool$ or nat, $[\![M]\!] = [\![V]\!] \in [\![\tau]\!] \implies M \Downarrow_\tau V$.

Theorem. *For all types* τ *and closed terms* $M_1, M_2 \in \mathrm{PCF}_\tau$, *if* $[\![M_1]\!]$ *and* $[\![M_2]\!]$ *are equal elements of the domain* $[\![\tau]\!]$, *then* $M_1 \cong_{\mathrm{ctx}} M_2 : \tau$.

Proof principle

$$\frac{[\![M_1]\!] = [\![M_2]\!]}{M_1 \cong_{\mathrm{ctx}} M_2}$$

Theorem. *For all types τ and closed terms $M_1, M_2 \in \mathrm{PCF}_\tau$, if $[\![M_1]\!]$ and $[\![M_2]\!]$ are equal elements of the domain $[\![\tau]\!]$, then $M_1 \cong_{\mathrm{ctx}} M_2 : \tau$.*

Proof. $M_1 \leq_{ctx} M_2 \overset{def}{\Leftrightarrow} (\forall \mathcal{C}[-].\ \mathcal{C}[M_1]\Downarrow V \Rightarrow \mathcal{C}[M_2]\Downarrow V)$

$$\mathcal{C}[M_1] \Downarrow_{nat} V \Rightarrow [\![\mathcal{C}[M_1]]\!] = [\![V]\!] \quad \text{(soundness)}$$

|| ~ by compositionality

$$\Rightarrow [\![\mathcal{C}[M_2]]\!] = [\![V]\!] \quad \text{(compositionality on } [\![M_1]\!] = [\![M_2]\!])$$

$$\Rightarrow \mathcal{C}[M_2] \Downarrow_{nat} V \quad \text{(adequacy)}$$

and symmetrically. □

$M_1 \cong_{ctx} M_2$ iff $M_1 \leq_{ctx} M_2$ & $M_2 \leq_{ctx} M_1$

Proof principle

$$\frac{[\![M_1]\!] \sqsubseteq [\![M_2]\!]}{M_1 \leq_{ctx} M_2}$$

value
$\wr$

$\mathcal{C}[M_1] \Downarrow_{nat} V \implies [\![\mathcal{C}[M_1]]\!] = [\![V]\!]$

$\implies [\![V]\!] \sqsubseteq [\![\mathcal{C}[M_2]]\!]$

in the flat domain of $\mathbb{N}$: $0 \quad 1 \ldots n \ldots$ over $\bot$

$\implies [\![\mathcal{C}[M_2]]\!] = [\![V]\!]$

$\implies \mathcal{C}[M_2] \Downarrow V$ $\square$

COMPOSITIONALITY:

$[\![M_1]\!] \sqsubseteq [\![M_2]\!] \longrightarrow [\![\mathcal{C}[M_1]]\!] \sqsubseteq [\![\mathcal{C}[M_2]]\!]$

PROOF PRINCIPLE:

$$\frac{\llbracket F \rrbracket \left(\llbracket P \rrbracket \right) \subseteq \llbracket P \rrbracket}{\dfrac{\mathit{fix}\left(\llbracket F \rrbracket \right) = \llbracket \mathit{fix}(F) \rrbracket \subseteq \llbracket P \rrbracket}{\mathit{fix}(F) \leq_{ctx} P}}$$

Proof principle

To prove

$$M_1 \cong_{\mathrm{ctx}} M_2 : \tau$$

it suffices to establish

$$[\![M_1]\!] = [\![M_2]\!] \text{ in } [\![\tau]\!]$$

? The proof principle is sound, but is it complete? That is, is equality in the denotational model also a necessary condition for contextual equivalence?

Topic 6

Denotational Semantics of PCF

Denotational semantics of PCF

To every typing judgement

$$\Gamma \vdash M : \tau$$

we associate a continuous function

$$[\![\Gamma \vdash M]\!] : [\![\Gamma]\!] \to [\![\tau]\!]$$

between domains.

Denotational semantics of PCF types

$$[\![nat]\!] \stackrel{\text{def}}{=} \mathbb{N}_\perp$$ (flat domain)

$$[\![bool]\!] \stackrel{\text{def}}{=} \mathbb{B}_\perp$$ (flat domain)

$$[\![\tau \to \tau']\!] \stackrel{\text{def}}{=} [\![\tau]\!] \to [\![\tau']\!]$$ (function domain).

where $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{B} = \{true, false\}$.

$\Gamma \equiv (x_1 \mapsto \tau_1, x_2 \mapsto \tau_2, \ldots, x_n \mapsto \tau_n) \equiv (x_1 : \tau_1, \ldots, x_n : \tau_n)$

Denotational semantics of PCF type environments

$[\![\Gamma]\!] \stackrel{\text{def}}{=} \prod_{x \in dom(\Gamma)} [\![\Gamma(x)]\!]$ (Γ-environments)

$= [\![\tau_1]\!] \times \cdots \times [\![\tau_n]\!]$

$\rho \in [\![\Gamma]\!]$ ~ domain of environments

$\rho = (d_1, \ldots, d_n)$ with $d_i \in [\![\tau_i]\!]$

Denotational semantics of PCF type environments

$$[\![\Gamma]\!] \stackrel{\text{def}}{=} \prod_{x \in dom(\Gamma)} [\![\Gamma(x)]\!] \qquad \text{(Γ-environments)}$$

$=$ the domain of partial functions ρ from variables to domains such that $dom(\rho) = dom(\Gamma)$ and $\rho(x) \in [\![\Gamma(x)]\!]$ for all $x \in dom(\Gamma)$

Example:

1. For the empty type environment $\emptyset$,

$$[\![\emptyset]\!] = \{\,\bot\,\}$$

where $\bot$ denotes the unique partial function with $dom(\bot) = \emptyset$.

2. $[\![\langle x \mapsto \tau \rangle]\!] = (\{\, x \,\} \to [\![\tau]\!]) \cong [\![\tau]\!]$

3.

$$
\begin{aligned}
& [\![\langle x_1 \mapsto \tau_1, \ldots, x_n \mapsto \tau_n \rangle]\!] \\
& \quad \cong \left(\{\, x_1 \,\} \to [\![\tau_1]\!]\right) \times \ldots \times \left(\{\, x_n \,\} \to [\![\tau_n]\!]\right) \\
& \quad \cong [\![\tau_1]\!] \times \ldots \times [\![\tau_n]\!]
\end{aligned}
$$

$$[\![[x_1 : \tau_1, \ldots, x_n : \tau_n]]\!] = [\![\tau_1]\!] \times \ldots \times [\![\tau_n]\!]$$

Aim: $\Gamma \vdash M : \tau \rightsquigarrow [\![\Gamma \vdash M]\!] : [\![\Gamma]\!] \to [\![\tau]\!]$ cont

Denotational semantics of PCF terms, I

$[\![\Gamma \vdash \mathbf{0}]\!] = \lambda \rho.\, 0 : [\![\Gamma]\!] \to [\![nat]\!]$

$$[\![\Gamma \vdash \mathbf{0}]\!](\rho) \stackrel{\text{def}}{=} 0 \in [\![nat]\!]$$

$$[\![\Gamma \vdash \mathbf{true}]\!](\rho) \stackrel{\text{def}}{=} true \in [\![bool]\!]$$

$$[\![\Gamma \vdash \mathbf{false}]\!](\rho) \stackrel{\text{def}}{=} false \in [\![bool]\!]$$

$\Gamma \equiv (x_1 : \tau_1, \ldots, x_n : \tau_n)$

Denotational semantics of PCF terms, I

$[\![\Gamma \vdash x_i : \tau_i]\!] : [\![\tau_1]\!] \times \cdots \times [\![\tau_n]\!] \longrightarrow [\![\tau_i]\!]$

$(d_1, \ldots, d_n) \longmapsto d_i$

i-th projection function

$$[\![\Gamma \vdash \mathbf{0}]\!](\rho) \stackrel{\text{def}}{=} 0 \in [\![nat]\!]$$

$$[\![\Gamma \vdash \mathbf{true}]\!](\rho) \stackrel{\text{def}}{=} true \in [\![bool]\!]$$

$$[\![\Gamma \vdash \mathbf{false}]\!](\rho) \stackrel{\text{def}}{=} false \in [\![bool]\!]$$

$$[\![\Gamma \vdash x]\!](\rho) \stackrel{\text{def}}{=} \rho(x) \in [\![\Gamma(x)]\!] \qquad (x \in dom(\Gamma))$$

Denotational semantics of PCF terms, II

$$[\![\Gamma \vdash \mathbf{succ}(M)]\!](\rho)$$

$$\stackrel{\text{def}}{=} \begin{cases} [\![\Gamma \vdash M]\!](\rho) + 1 & \text{if } [\![\Gamma \vdash M]\!](\rho) \neq \bot \\ \bot & \text{if } [\![\Gamma \vdash M]\!](\rho) = \bot \end{cases}$$

$$\mathbb{N}_\bot \to \mathbb{N}_\bot : n \mapsto \begin{cases} n+1 & \text{otw} \\ \bot & \text{if } n = \bot \end{cases}$$

Denotational semantics of PCF terms, II

$$[\![\Gamma \vdash \mathbf{succ}(M)]\!](\rho)$$

$$\stackrel{\text{def}}{=} \begin{cases} [\![\Gamma \vdash M]\!](\rho) + 1 & \text{if } [\![\Gamma \vdash M]\!](\rho) \neq \bot \\ \bot & \text{if } [\![\Gamma \vdash M]\!](\rho) = \bot \end{cases}$$

$$[\![\Gamma \vdash \mathbf{pred}(M)]\!](\rho)$$

$$\stackrel{\text{def}}{=} \begin{cases} [\![\Gamma \vdash M]\!](\rho) - 1 & \text{if } [\![\Gamma \vdash M]\!](\rho) > 0 \\ \bot & \text{if } [\![\Gamma \vdash M]\!](\rho) = 0, \bot \end{cases}$$

$$\mathbb{N}_\bot \longrightarrow \mathbb{N}_\bot : n \longmapsto \begin{cases} n-1 & \text{otw} \\ \bot & \text{if } n = 0, \bot \end{cases}$$

Denotational semantics of PCF terms, II

$$[\![\Gamma \vdash \mathbf{succ}(M)]\!](\rho)$$

$$\stackrel{\text{def}}{=} \begin{cases} [\![\Gamma \vdash M]\!](\rho) + 1 & \text{if } [\![\Gamma \vdash M]\!](\rho) \neq \bot \\ \bot & \text{if } [\![\Gamma \vdash M]\!](\rho) = \bot \end{cases}$$

$$[\![\Gamma \vdash \mathbf{pred}(M)]\!](\rho)$$

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$$[\![\Gamma \vdash \mathbf{zero}(M)]\!](\rho) \stackrel{\text{def}}{=} \begin{cases} \mathit{true} & \text{if } [\![\Gamma \vdash M]\!](\rho) = 0 \\ \mathit{false} & \text{if } [\![\Gamma \vdash M]\!](\rho) > 0 \\ \bot & \text{if } [\![\Gamma \vdash M]\!](\rho) = \bot \end{cases}$$

$$\mathbb{N}_\bot \to \mathbb{B}_\bot : \begin{cases} \bot \mapsto \bot \\ 0 \mapsto \mathit{true} \\ n \mapsto \mathit{false} \end{cases}$$

Denotational semantics of PCF terms, III

$$[\![\Gamma \vdash \mathbf{if}\ M_1\ \mathbf{then}\ M_2\ \mathbf{else}\ M_3]\!](\rho)$$

$$\stackrel{\text{def}}{=} \begin{cases} [\![\Gamma \vdash M_2]\!](\rho) & \text{if } [\![\Gamma \vdash M_1]\!](\rho) = \mathit{true} \\ [\![\Gamma \vdash M_3]\!](\rho) & \text{if } [\![\Gamma \vdash M_1]\!](\rho) = \mathit{false} \\ \bot & \text{if } [\![\Gamma \vdash M_1]\!](\rho) = \bot \end{cases}$$

$$B_\perp \times D \times D \to D$$

$$\begin{array}{llll} \bot & d & d' & \mapsto \bot \\ true & d & d' & \mapsto d \\ false & d & d' & \mapsto d' \end{array}$$

Denotational semantics of PCF terms, III

$$[\![\Gamma \vdash \mathbf{if}\ M_1\ \mathbf{then}\ M_2\ \mathbf{else}\ M_3]\!](\rho)$$

$$\stackrel{\text{def}}{=} \begin{cases} [\![\Gamma \vdash M_2]\!](\rho) & \text{if } [\![\Gamma \vdash M_1]\!](\rho) = \mathit{true} \\ [\![\Gamma \vdash M_3]\!](\rho) & \text{if } [\![\Gamma \vdash M_1]\!](\rho) = \mathit{false} \\ \bot & \text{if } [\![\Gamma \vdash M_1]\!](\rho) = \bot \end{cases}$$

$$[\![\Gamma \vdash M_1\, M_2]\!](\rho) \stackrel{\text{def}}{=} \big([\![\Gamma \vdash M_1]\!](\rho)\big)\,([\![\Gamma \vdash M_2]\!](\rho))$$

application

evaluation

$$[\![\Gamma \vdash M_1\ M_2 : \tau_1]\!] : [\![\Gamma]\!] \to [\![\tau_1]\!]$$

$$\Gamma \vdash M_1 : \tau_2 \to \tau_1$$

$$\Gamma \vdash M_2 : \tau_2$$

(fn ML
fn f => fn x => f(x))

$$[\![\Gamma \vdash M_1]\!] : [\![\Gamma]\!] \to ([\![\tau_2]\!] \to [\![\tau_1]\!])$$

$$[\![\Gamma \vdash M_2]\!] : [\![\Gamma]\!] \to [\![\tau_2]\!]$$

$$(D_2 \to D_1) \times D_2 \to D_1 : f, x \mapsto f(x)$$

continuous

EXERCISE

$$\frac{\Gamma[x \mapsto \tau] \vdash M : \tau'}{\Gamma \vdash \mathbf{fn}\, x : \tau .\, M : \tau \to \tau'} \quad \rightsquigarrow \quad [\![\Gamma]\!] \times [\![\tau]\!] \longrightarrow [\![\tau']\!]$$

$$[\![\Gamma]\!] \longrightarrow ([\![\tau]\!] \to [\![\tau']\!])$$

Currying

Denotational semantics of PCF terms, IV

$$[\![\Gamma \vdash \mathbf{fn}\, x : \tau \,.\, M]\!](\rho) \qquad (x \notin dom(\Gamma))$$

$$\stackrel{\text{def}}{=} \lambda d \in [\![\tau]\!] \,.\, [\![\Gamma[x \mapsto \tau] \vdash M]\!](\rho[x \mapsto d])$$

NB: $\rho[x \mapsto d] \in [\![\Gamma[x \mapsto \tau]]\!]$ is the function mapping x to $d \in [\![\tau]\!]$ and otherwise acting like ρ.

If $f : D \times E \to F$ cont. then $\hat{f} : D \to (E \to F)$
$d \mapsto \lambda e . f(d,e)$
is cont.

EXERCISE

Denotational semantics of PCF terms, V

$$[\![\Gamma \vdash \mathbf{fix}(M)]\!](\rho) \stackrel{\text{def}}{=} \mathit{fix}([\![\Gamma \vdash M]\!](\rho))$$

$$\left(\mathit{fix} \circ [\![\Gamma \vdash M]\!]\right)(\rho)$$

Recall that fix is the function assigning least fixed points to continuous functions.

Denotational semantics of PCF

Proposition. *For all typing judgements* $\Gamma \vdash M : \tau$, *the denotation*

$$[\![\Gamma \vdash M]\!] : [\![\Gamma]\!] \to [\![\tau]\!]$$

is a well-defined continous function.