Topic 3

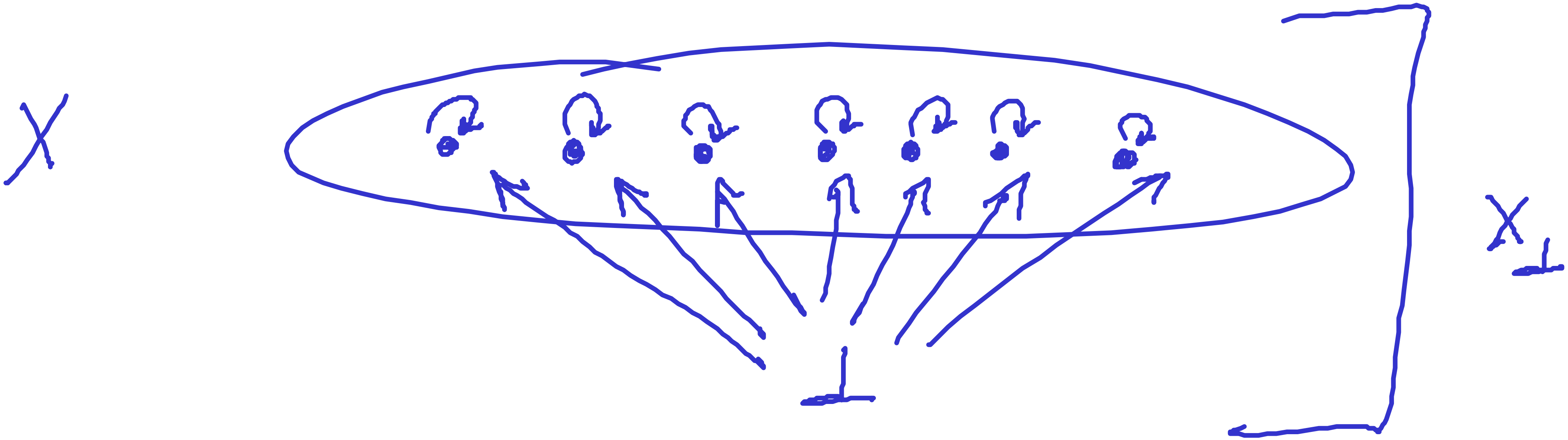
Constructions on Domains

Discrete cpo's and flat domains

For any set X, the relation of equality

$$x \sqsubseteq x' \overset{\text{def}}{\Leftrightarrow} x = x' \qquad (x, x' \in X)$$

makes $(X, \sqsubseteq)$ into a cpo, called the discrete cpo with underlying set X.



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Let $X_\perp \overset{\text{def}}{=} X \cup \{\perp\}$, where $\perp$ is some element not in X. Then

$$d \sqsubseteq d' \overset{\text{def}}{\Leftrightarrow} (d = d') \vee (d = \perp) \qquad (d, d' \in X_\perp)$$

makes $(X_\perp, \sqsubseteq)$ into a domain (with least element $\perp$), called the flat domain determined by X.

Product domains

(In ML, the type constructor $*$)

Given D_1 and D_2 domains

define $D_1 \times D_2$

as follows

- underlying set

$$\{ (d_1, d_2) \mid d_1 \in D_1 \ \& \ d_2 \in D_2 \}$$

- with order

$$(d_1, d_2) \sqsubseteq (d_1', d_2')$$

iff

$$d_1 \sqsubseteq_{D_1} d_1' \quad \& \quad d_2 \sqsubseteq_{D_2} d_2'.$$

pointwise

- least element

$$\perp_{D_1 \times D_2} = (\perp_{D_1}, \perp_{D_2})$$

- lubs of countable chains.

a chain in $D_1 \times D_2$ looks like

$$(x_0, y_0) \sqsubseteq (x_1, y_1) \sqsubseteq \cdots \sqsubseteq (x_n, y_n) \sqsubseteq \cdots$$

which gives

$$x_0 \sqsubseteq x_1 \sqsubseteq \cdots \sqsubseteq x_n \sqsubseteq \cdots \quad \text{in } D_1$$

$$y_0 \sqsubseteq y_1 \sqsubseteq \cdots \sqsubseteq y_n \sqsubseteq \cdots \quad \text{in } D_2$$

with lubs $\bigsqcup_n x_n$ in D_1

and $\bigsqcup_n y_n$ in D_2.

So we have

$$\left(\bigsqcup_n x_n, \bigsqcup_n y_n\right) \text{ in } D_1 \times D_2$$

<u>Claim</u>

is the lub of

$$(x_0, y_0) \sqsubseteq \ldots \sqsubseteq (x_n, y_n) \sqsubseteq \ldots$$

That is

$$\bigsqcup_n (x_n, y_n) = \left(\bigsqcup_n x_n, \bigsqcup_n y_n\right).$$

Binary product of cpo's and domains

The product of two cpo's $(D_1, \sqsubseteq_1)$ and $(D_2, \sqsubseteq_2)$ has underlying set

$$D_1 \times D_2 = \{(d_1, d_2) \mid d_1 \in D_1 \;\&\; d_2 \in D_2\}$$

and partial order $\sqsubseteq$ defined by

$$(d_1, d_2) \sqsubseteq (d'_1, d'_2) \;\stackrel{\text{def}}{\Leftrightarrow}\; d_1 \sqsubseteq_1 d'_1 \;\&\; d_2 \sqsubseteq_2 d'_2 \ .$$

$$\frac{(x_1, x_2) \sqsubseteq (y_1, y_2)}{x_1 \sqsubseteq_1 y_1 \qquad x_2 \sqsubseteq_2 y_2}$$

Lubs of chains are calculated componentwise:

$$\bigsqcup_{n \geq 0} (d_{1,n}, d_{2,n}) = (\bigsqcup_{i \geq 0} d_{1,i}, \bigsqcup_{j \geq 0} d_{2,j}) \ .$$

If $(D_1, \sqsubseteq_1)$ and $(D_2, \sqsubseteq_2)$ are domains so is $(D_1 \times D_2, \sqsubseteq)$ and $\perp_{D_1 \times D_2} = (\perp_{D_1}, \perp_{D_2})$.

Let D, E, F be domains and

$f : (D \times E) \to F$ continuous.

monotone: $(d,e) \sqsubseteq (d',e') \Rightarrow f(d,e) \sqsubseteq f(d',e')$

cont. $f(\sqcup_n (d_n, e_n)) = \sqcup_n f(d_n, e_n)$

Iff f is continuous in D and E separately; i.e.,

$\forall d \in D, \quad f(d, -) = \lambda e.\, f(d,e)$

$\forall e \in E, \quad f(-, e) = \lambda d.\, f(d,e)$

cont.

f cont in D and E arguments

<u>RTP</u>: $f\left(\bigsqcup_n (d_n, e_n)\right) \overset{?}{=} \bigsqcup_n f(d_n, e_n)$

$$\begin{aligned}
&\parallel \\
&f\left(\bigsqcup_n d_n, \bigsqcup_n e_n\right) \\
&\parallel \quad \text{cont on D argument} \\
&\bigsqcup_n f\left(d_n, \bigsqcup_n e_n\right) \\
&\parallel \quad \text{cont on E argument} \\
&\bigsqcup_n \bigsqcup_m f(d_n, e_m)
\end{aligned}$$

= by Diag Lemma.

Continuous functions of two arguments

Proposition. *Let* D, E, F *be cpo's. A function* $f : (D \times E) \to F$ *is monotone if and only if it is monotone in each argument separately:*

$$\forall d, d' \in D, e \in E.\, d \sqsubseteq d' \;\Rightarrow\; f(d,e) \sqsubseteq f(d',e)$$

$$\forall d \in D, e, e' \in E.\, e \sqsubseteq e' \;\Rightarrow\; f(d,e) \sqsubseteq f(d,e').$$

Moreover, it is continuous if and only if it preserves lubs of chains in each argument separately:

$$f(\bigsqcup_{m \geq 0} d_m\, , \, e) = \bigsqcup_{m \geq 0} f(d_m, e)$$

$$f(d\, , \bigsqcup_{n \geq 0} e_n) = \bigsqcup_{n \geq 0} f(d, e_n).$$

- A couple of derived rules:

$$\frac{x \sqsubseteq x' \qquad y \sqsubseteq y'}{f(x,y) \sqsubseteq f(x',y')} \quad (f \text{ monotone})$$

$$\frac{}{f(\bigsqcup_m x_m, \bigsqcup_n y_n) = \bigsqcup_k f(x_k, y_k)} \quad (f \text{ cont.})$$

Given D and E domains define the function domain

$$(D \to E)$$

as follows

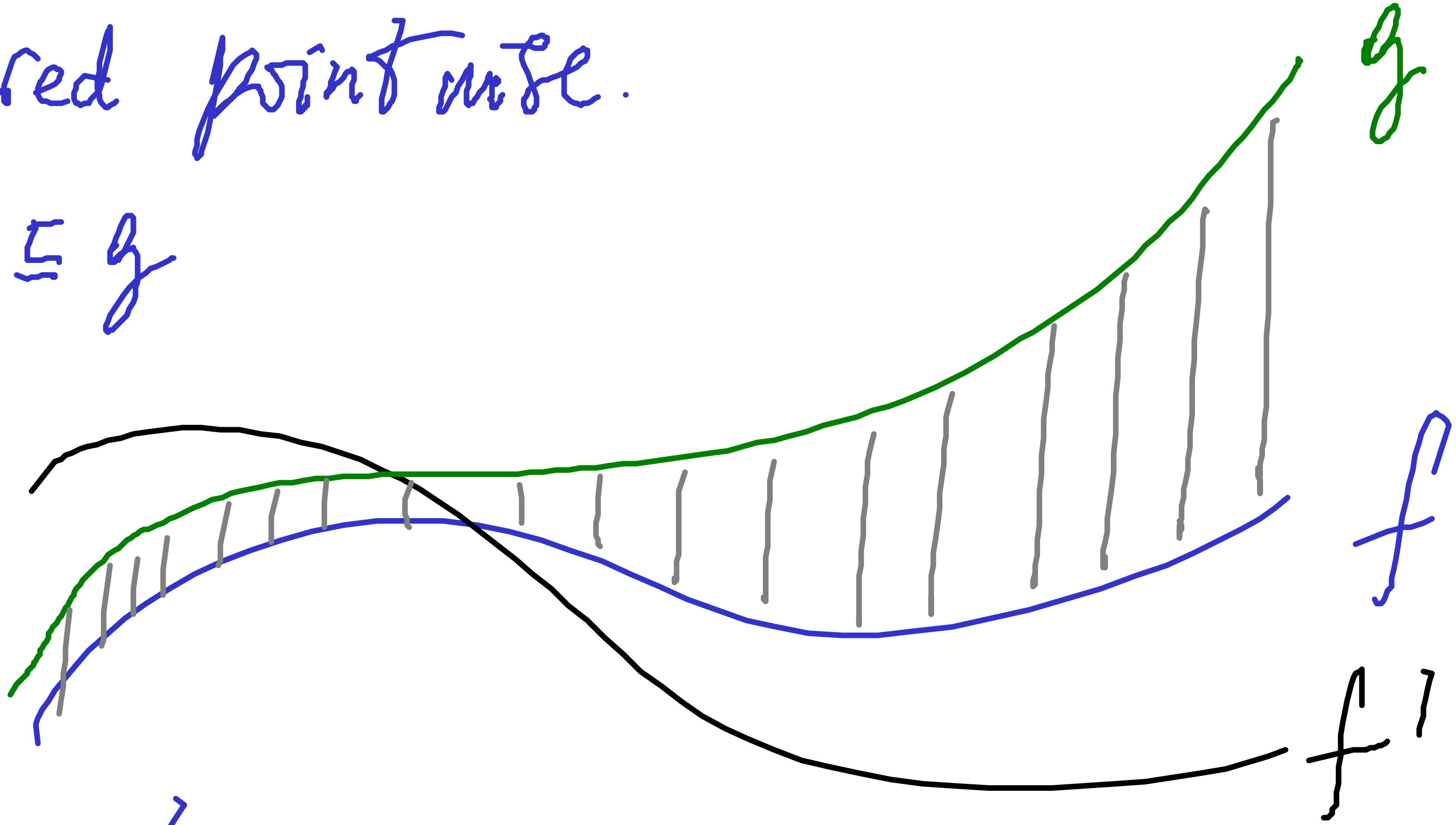
- underlying set

$$\{ f \mid f \text{ is a cont. from } D \text{ to } E \}$$

In ML, like the $\to$ type constructor

- ordered pointwise

$f \sqsubseteq g$



$f \not\sqsubseteq f'$, $g \not\sqsubseteq f'$, $f' \not\sqsubseteq f$, ...

$\forall d \in D.\ f(d) \sqsubseteq_E g(d)$

- least element

$$\perp_{(D\to E)} \in (D\to E)$$

$$\| $$

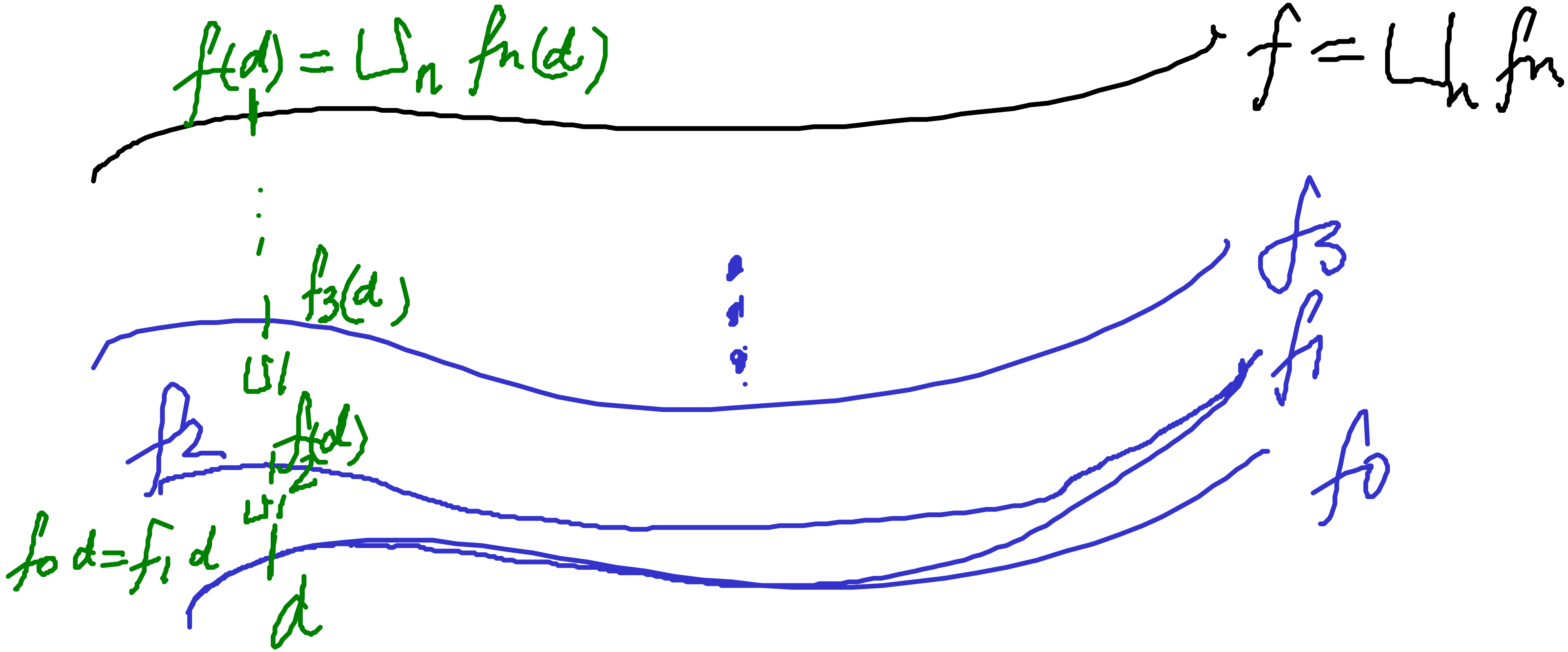
$$\lambda d.\perp_E \sqsubseteq f \in (D\to E)$$

because

$$\forall d \quad \perp_{D\to E}(d) = \perp_E \sqsubseteq f(d).$$

- lubs

$$f_0 \sqsubseteq f_1 \sqsubseteq \cdots \sqsubseteq f_n \sqsubseteq \cdots \quad \text{in } (D\to E)$$



To define such f we need define $f(d)$ $\forall d$

$$f(d) \stackrel{def}{=} \bigsqcup_n f_n(d)$$

$\rightsquigarrow$ well defined because $f_n(d)$ is a chain.

$f_n \sqsubseteq f$ iff $f_n(d) \sqsubseteq f(d)$ ✓

$f_n \sqsubseteq g \Rightarrow f \sqsubseteq g$

We need show

- f is cont.
- f is a lub of the f_n.

$$d \sqsubseteq d' \Rightarrow \bigsqcup_n f_n(d) \sqsubseteq \bigsqcup_n f_n(d') \quad ✓$$

$$d_0 \sqsubseteq d_1 \sqsubseteq \cdots \sqsubseteq d_n \sqsubseteq \cdots$$

$$f\left(\bigsqcup_n d_n\right) \overset{?}{=} \bigsqcup_n f(d_n)$$

$$\bigsqcup_m f_m\left(\bigsqcup_n d_n\right) \qquad\qquad \bigsqcup_n \bigsqcup_m f_m(d_n)$$

$$= \bigsqcup_m \bigsqcup_n f_m(d_n) = \text{ by Diag Lemma}$$

<u>Exercise</u> Let P be a poset and D be a domain. Then define $(P \Rightarrow D)$ to be the set of monotone functions from P to D ordered pointwise.

<u>Claim</u>: $(P \Rightarrow D)$ is a domain.

Function cpo's and domains

Given cpo's $(D, \sqsubseteq_D)$ and $(E, \sqsubseteq_E)$, the function cpo $(D \to E, \sqsubseteq)$ has underlying set

$$(D \to E) \stackrel{\text{def}}{=} \{f \mid f : D \to E \text{ is a \textit{continuous} function}\}$$

and partial order: $f \sqsubseteq f' \stackrel{\text{def}}{\Leftrightarrow} \forall d \in D \,.\, f(d) \sqsubseteq_E f'(d)$.

- A derived rule:

$$\frac{f \sqsubseteq_{(D \to E)} g \qquad x \sqsubseteq_D y}{f(x) \sqsubseteq g(y)}$$

Lubs of chains are calculated 'argumentwise' (using lubs in E):

$$\bigsqcup_{n\geq 0} f_n \;=\; \lambda d \in D.\, \bigsqcup_{n\geq 0} f_n(d)\;\;.$$

- A derived rule:

$$\frac{}{(\bigsqcup_n f_n)(\bigsqcup_m x_m) = \bigsqcup_k f_k(x_k)}$$

If E is a domain, then so is $D \to E$ and $\perp_{D\to E}(d) = \perp_E$, all $d \in D$.

In ML, $o : (\beta \to \gamma) * (\alpha \to \beta) \to (\alpha \to \gamma)$

Continuity of composition

For cpo's D, E, F, the composition function

Exercise

$$\circ : \big((E \to F) \times (D \to E)\big) \longrightarrow (D \to F)$$

defined by setting, for all $f \in (D \to E)$ and $g \in (E \to F)$,

$$g \circ f \;=\; \lambda d \in D.\, g\big(f(d)\big)$$

is continuous.