__Problem__

$$\bigsqcup_m d_{m,0} \sqsubseteq \bigsqcup_m d_{m,1} \sqsubseteq \cdots \bigsqcup_n \Big(\bigsqcup_m d_{m,n}\Big) = \bigsqcup_m \Big(\bigsqcup_n d_{m,n}\Big)$$

$$\bigsqcup_m d_{m,0} \sqsupseteq \cdots$$

$$\bigsqcup_n \Big(\bigsqcup_m d_{m,n}\Big) = \bigsqcup_k d_{k,k} = \bigsqcup_m \Big(\bigsqcup_n d_{m,n}\Big)$$

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
d_{n,0} & \sqsubseteq & d_{n,1} & \sqsubseteq & d_{n,2} & \sqsubseteq & \cdots \\
\sqcup\!\!\mid & & \sqcup\!\!\mid & & \sqcup\!\!\mid & & \\
\vdots & & \vdots & & \vdots & \sqsubseteq & \cdots \\
\sqcup\!\!\mid & & \sqcup\!\!\mid & \sqsubseteq & \sqcup\!\!\mid & & \\
d_{1,0} & \sqsubseteq & d_{1,1} & \sqsubseteq & d_{1,2} & \sqsubseteq & \cdots \\
\sqcup\!\!\mid & \sqsubseteq & \sqcup\!\!\mid & & \sqcup\!\!\mid & & \\
d_{0,0} & \sqsubseteq & d_{0,1} & \sqsubseteq & d_{0,2} & \sqsubseteq & \cdots
\end{array}
\qquad
\begin{array}{c}
\vdots \\
\sqcup\!\!\mid \\
\bigsqcup_n d_{2,n} \\
\sqcup\!\!\mid \\
\bigsqcup_n d_{1,n} \\
\sqcup\!\!\mid \\
\bigsqcup_n d_{0,n}
\end{array}$$

Application

$$fn\ f \Rightarrow fn\ x \Rightarrow f(x)$$

$$\lambda f . \lambda x . f(x)$$

We will show such functions to be continuous, using the diagonalisation lemma.

PROOF: $\bigsqcup_k d_{kk} = \bigsqcup_n \left(\bigsqcup_m d_{m,n} \right)$

(1) $\bigsqcup_k d_{kk} \sqsubseteq \bigsqcup_n \left(\bigsqcup_m d_{m,n} \right)$

$$\frac{d_{m,n} \sqsubseteq \bigsqcup_m d_{m,n}}{d_{nn} \sqsubseteq \bigsqcup_n d_{m,n} \sqsubseteq \bigsqcup_n \bigsqcup_m d_{mn}}$$

$$\frac{\forall k \quad d_{kk} \sqsubseteq \bigsqcup_n \left(\bigsqcup_m d_{m,n} \right)}{\bigsqcup_k d_{kk} \sqsubseteq \bigsqcup_n \left(\bigsqcup_m d_{m,n} \right)}$$

(2) $\bigsqcup_m \bigsqcup_n d_{m,n} \sqsubseteq \bigsqcup_k d_{kk}$

$$\forall m\ \forall n \quad d_{m,n} \sqsubseteq d_{\max(m,n),\max(m,n)} \sqsubseteq \bigsqcup_k d_{kk}$$

$$\forall m\ \forall n \quad d_{m,n} \sqsubseteq \bigsqcup_k d_{k,k}$$

$$\forall m \quad \bigsqcup_n d_{m,n} \sqsubseteq \bigsqcup_k d_{kk}$$

$$\bigsqcup_m \bigsqcup_n d_{m,n} \sqsubseteq \bigsqcup_k d_{kk}$$

Diagonalising a double chain

Lemma. *Let D be a cpo. Suppose that the doubly-indexed family of elements $d_{m,n} \in D$ ($m, n \geq 0$) satisfies*

$$m \leq m' \ \&\ n \leq n' \ \Rightarrow\ d_{m,n} \sqsubseteq d_{m',n'}. \qquad (\dagger)$$

Then

$$\bigsqcup_{n\geq 0} d_{0,n} \ \sqsubseteq\ \bigsqcup_{n\geq 0} d_{1,n} \ \sqsubseteq\ \bigsqcup_{n\geq 0} d_{2,n} \ \sqsubseteq\ \ldots$$

and

$$\bigsqcup_{m\geq 0} d_{m,0} \ \sqsubseteq\ \bigsqcup_{m\geq 0} d_{m,1} \ \sqsubseteq\ \bigsqcup_{m\geq 0} d_{m,3} \ \sqsubseteq\ \ldots$$

Moreover

$$\bigsqcup_{m\geq 0}\left(\bigsqcup_{n\geq 0} d_{m,n}\right) = \bigsqcup_{k\geq 0} d_{k,k} = \bigsqcup_{n\geq 0}\left(\bigsqcup_{m\geq 0} d_{m,n}\right) .$$

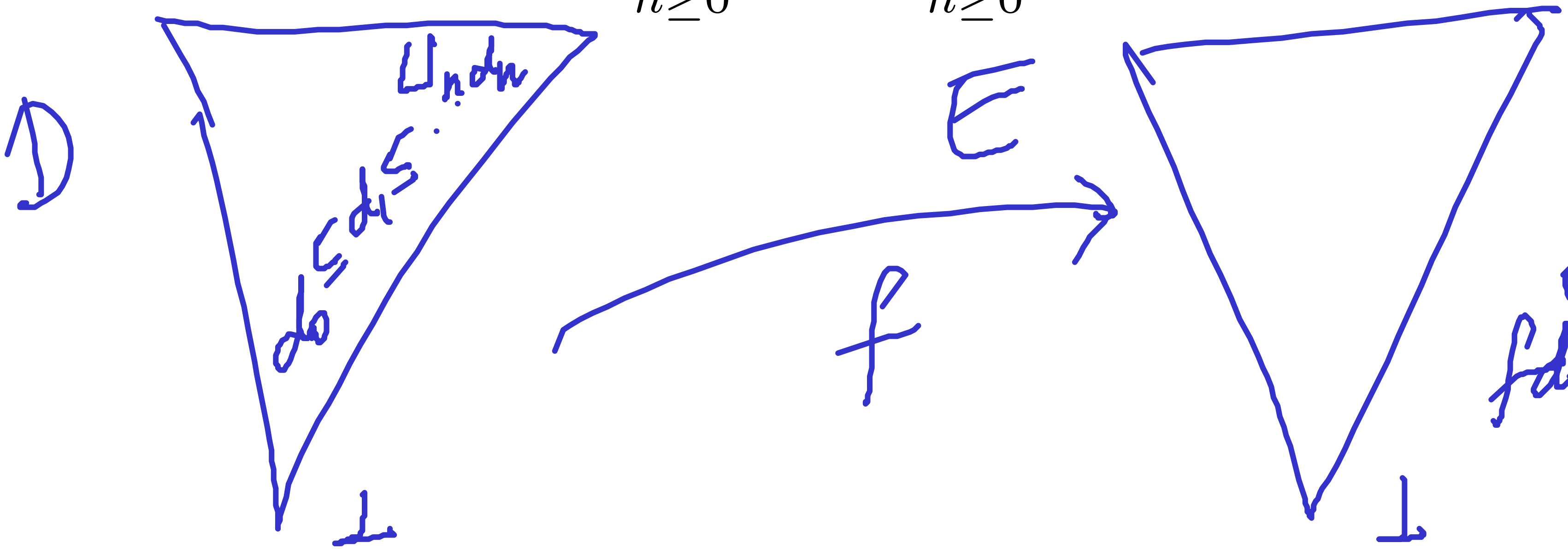
Continuity and strictness

- If D and E are cpo's, the function f is continuous iff
  1. it is monotone, and
  2. it preserves lubs of chains, *i.e.* for all chains $d_0 \sqsubseteq d_1 \sqsubseteq \ldots$ in D, it is the case that

$$f(\bigsqcup_{n\geq 0} d_n) = \bigsqcup_{n\geq 0} f(d_n) \quad \text{in } E.$$

asserts =

$f(\bigsqcup_n d_n) = \bigsqcup_n f(d_n)$



$fd_0 \sqsubseteq fd_1 \sqsubseteq \ldots$

Continuity and strictness

- If D and E are cpo's, the function f is continuous iff

  1. it is monotone, and

  2. it preserves lubs of chains, *i.e.* for all chains $d_0 \sqsubseteq d_1 \sqsubseteq \dots$ in D, it is the case that

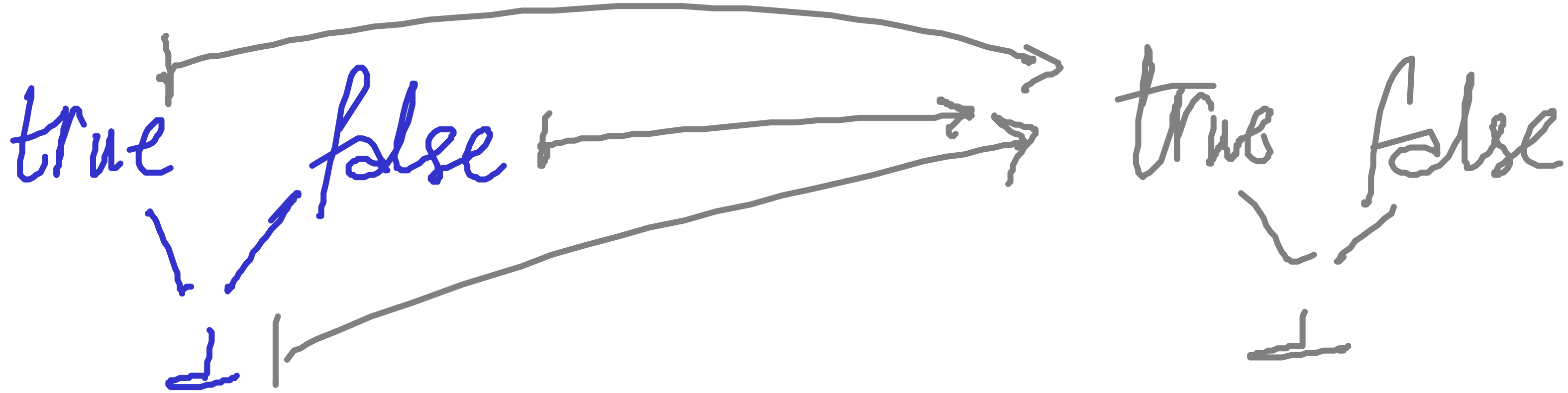
  $$f(\bigsqcup_{n \geq 0} d_n) = \bigsqcup_{n \geq 0} f(d_n) \quad \text{in } E.$$

- If D and E have least elements, then the function f is strict iff $f(\bot) = \bot$.

A non-strict function in call-by-name is $\lambda x.0$

domain of booleans

$\mathbb{B}$ =



f cont. function $\mathbb{B} \to \mathbb{B}$

iff it is monotone

(1) $f(\perp) \neq \perp \Rightarrow f(\text{true}) = f(\perp) = f(\text{false})$

Eg the function $\{\perp \mapsto \text{true}, \text{true} \mapsto \text{true}, \text{false} \mapsto \text{false}\}$

is not monotone

RECAP partial orders ⇝ pre fixed point fix

complete

Tarski's Fixed Point Theorem

Let $f : D \to D$ be a continuous function on a domain D. Then

- f possesses a least pre-fixed point, given by

$$\mathit{fix}(f) = \bigsqcup_{n \geq 0} f^n(\perp).$$

- Moreover, $\mathit{fix}(f)$ is a fixed point of f, *i.e.* satisfies $f\big(\mathit{fix}(f)\big) = \mathit{fix}(f)$, and hence is the least fixed point of f.

chain

$\perp \sqsubseteq f\perp \Rightarrow f\perp \sqsubseteq f(f\perp) = f^2(\perp) \ldots$

$\bigsqcup_n f^n(\bot)$ is a least prefixed point.

(1) $f\left(\bigsqcup_{n\geq 0} f^n(\bot)\right) \overset{\text{cont}}{\overset{\vee}{=}} \bigsqcup_{n\geq 0} f^{n+1}(\bot)$

$= \bigsqcup_{n\geq 1} f^n(\bot)$

$= \bigsqcup_{n\geq 0} f^n(\bot)$.

$\bot \sqsubseteq f\bot \sqsubseteq$ $\quad f^n\bot \sqsubseteq \text{---} \text{---}$

$f(\bot) \sqsubseteq f^2(\bot) \sqsubseteq$ $\quad f^{n+1}(\bot) \sqsubseteq \text{- - - -}$

(2) Let x be such that $\boxed{f(x) \sqsubseteq x}$

We need show $\bigsqcup_n f^n(\bot) \sqsubseteq x$

$$\frac{\bot \sqsubseteq x}{f(\bot) \sqsubseteq f(x) \sqsubseteq x} \text{ mon}$$

$\leadsto^{\text{induction}}$ $f^n(\bot) \sqsubseteq x$

$$\frac{\forall n \quad f^n(\bot) \sqsubseteq x}{\bigsqcup_n f^n(\bot) \sqsubseteq x}$$

$[\![\textbf{while } B \textbf{ do } C]\!]$

$[\![\textbf{while } B \textbf{ do } C]\!]$

NB: Is continuous

$= \mathit{fix}(f_{[\![B]\!],[\![C]\!]})$

$= \bigsqcup_{n \geq 0} {f_{[\![B]\!],[\![C]\!]}}^{n}(\bot)$

$$= \lambda s \in \mathit{State}. \begin{cases} [\![C]\!]^{k}(s) & \text{if } k \geq 0 \text{ is such that } [\![B]\!]([\![C]\!]^{k}(s)) = \mathit{false} \\ & \text{and } [\![B]\!]([\![C]\!]^{i}(s)) = \mathit{true} \text{ for all } 0 \leq i < k \\ \\ \text{undefined} & \text{if } [\![B]\!]([\![C]\!]^{i}(s)) = \mathit{true} \text{ for all } i \geq 0 \end{cases}$$

(1) $$[\![\text{while } B \text{ do } C]\!] = \underline{fix}\left(f_{[\![B]\!],[\![C]\!]}\right)$$

$$\Rightarrow f_{[\![B]\!],[\![C]\!]}\left([\![\text{while } B \text{ do } C]\!]\right)$$

$$= [\![\text{while } B \text{ do } C]\!]$$

(2) PROOF TECHNIQUE (eg. for program transformation)

$$\frac{f_{[\![B]\!],[\![C]\!]}\left([\![P]\!]\right) \sqsubseteq [\![P]\!]}{[\![\text{while } B \text{ do } C]\!] \sqsubseteq [\![P]\!]}$$

$$\frac{f(x) \sqsubseteq x}{\underline{fix}(f) \sqsubseteq x}$$

ML data structure

bool, int, ...

product (*)

function types

datatypes — enumerated

inductive (eg. trees, lists)

recursive

Topic 3

Constructions on Domains

model of λ calculus

$\sim \sqrt{}$ datatype$_D$ = fold of $D \to D$