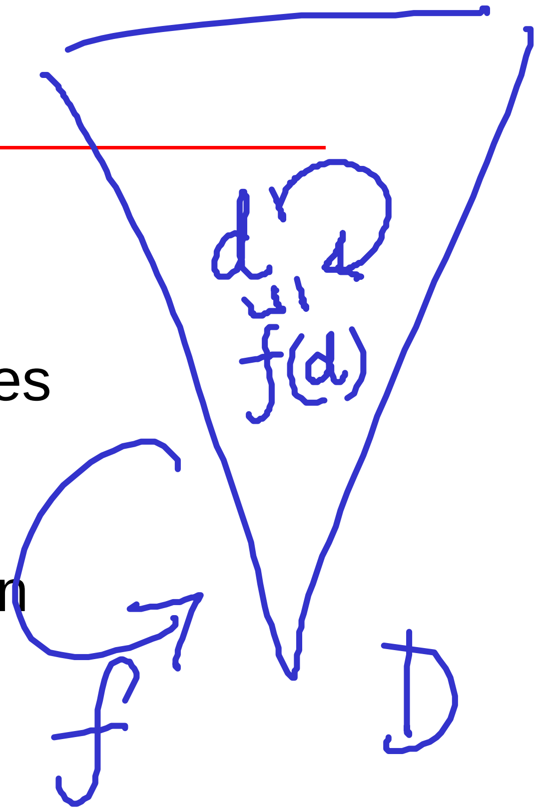
Pre-fixed points



Let D be a poset and $f : D \to D$ be a function.

An element $d \in D$ is a pre-fixed point of f if it satisfies $f(d) \sqsubseteq d$.

The *least pre-fixed point* of f, if it exists, will be written

$$\boxed{\mathit{fix}(f)}$$

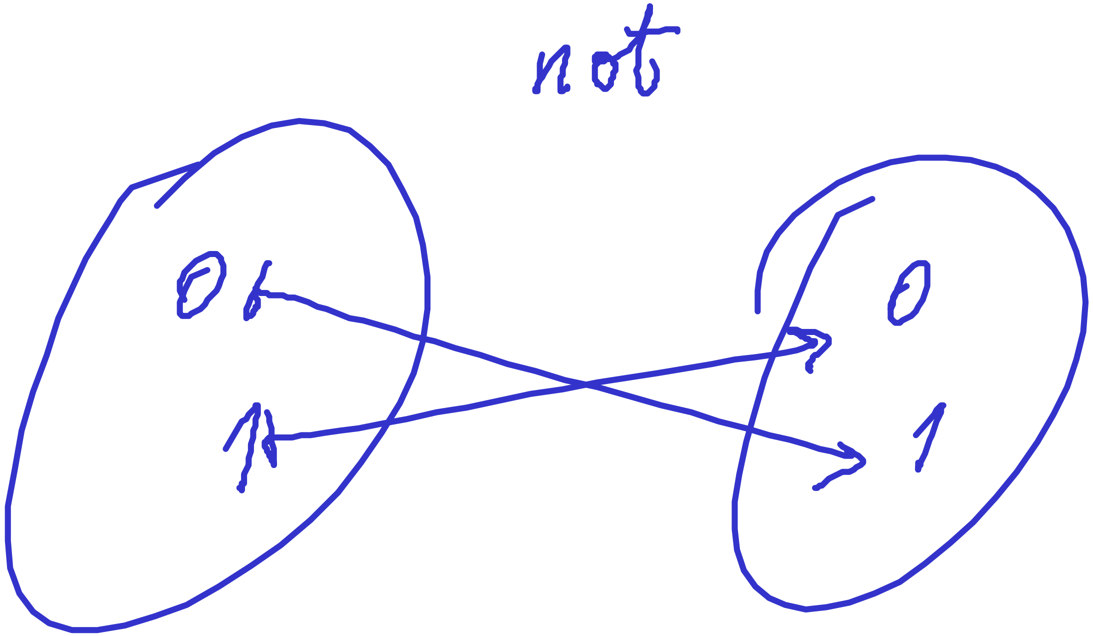
It is thus (uniquely) specified by the two properties:

$$f(\mathit{fix}(f)) \sqsubseteq \mathit{fix}(f) \qquad \text{(lfp1)}$$

$$\forall d \in D.\ f(d) \sqsubseteq d \ \Rightarrow\ \mathit{fix}(f) \sqsubseteq d. \qquad \text{(lfp2)}$$

$\boxed{?}$ Does any monotone function on a poset has pre-fixed points?

No:



Proof principle

1.

$$\frac{}{f(\mathit{fix}(f)) \sqsubseteq \mathit{fix}(f)}$$

2. Let D be a poset and let $f : D \to D$ be a function with a least pre-fixed point $\mathit{fix}(f) \in D$.

   For all $x \in D$, to prove that $\mathit{fix}(f) \sqsubseteq x$ it is enough to establish that $f(x) \sqsubseteq x$.

$$\frac{f(x) \sqsubseteq x}{\mathit{fix}(f) \sqsubseteq x}$$

f monotone

$\boxed{?}\ f(\mathit{fix}\, f) = \mathit{fix}(f)$?

$$\frac{x \sqsubseteq y}{fx \sqsubseteq fy}$$

Least pre-fixed points are fixed points

If it exists, the least pre-fixed point of a mononote function on a partial order is necessarily a fixed point.

$$\dfrac{\dfrac{f(\mathit{fix}\, f) \sqsubseteq \mathit{fix}\, f}{f(f(\mathit{fix}\, f)) \sqsubseteq f(\mathit{fix}\, f)}}{\mathit{fix}(f) \sqsubseteq f(\mathit{fix}\, f)}$$

$$\frac{}{f(\mathit{fix}\, f) \sqsubseteq \mathit{fix}\, f}$$

$$f(\mathit{fix}\, f) = \mathit{fix}\, f$$

Thesis*

All domains of computation are
complete partial orders with a least element.

they provide a notion of passage to the limit.

d $\qquad$ f monotone $\qquad$ $f(d) = e$

continuity

Thesis*

$d_0 \sqcup d_1 \sqcup d_2 \sqcup \cdots \qquad f(d_0) \sqcup f(d_1) \sqcup \cdots$

All domains of computation are
complete partial orders with a least element.

All computable functions are
continuous.

monotone + preservation property

Cpo's and domains

A chain complete poset, or cpo for short, is a poset $(D, \sqsubseteq)$ in which all countable increasing chains $d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \ldots$ have least upper bounds, $\bigsqcup_{n \geq 0} d_n$:

$$\forall m \geq 0 \,.\, d_m \sqsubseteq \bigsqcup_{n \geq 0} d_n \tag{lub1}$$

$$\forall d \in D \,.\, (\forall m \geq 0 \,.\, d_m \sqsubseteq d) \;\Rightarrow\; \bigsqcup_{n \geq 0} d_n \sqsubseteq d. \tag{lub2}$$

A domain is a cpo that possesses a least element, $\bot$:

$$\forall d \in D \,.\, \bot \sqsubseteq d.$$

denoted $\bigsqcup_{n\geq 0} d_n$

Lubs = passage to the limit

Given a countable chain

$$d_0 \sqsubseteq d_1 \sqsubseteq \dots \sqsubseteq d_n \sqsubseteq \dots \qquad (n \in \mathbb{N})$$

we are interested in its lub, defined by

(1) $\forall i \in \mathbb{N}. \quad d_i \sqsubseteq \bigsqcup_{n\geq 0} d_n$

(2) $\forall x \ (\forall i \ d_i \sqsubseteq x) \Rightarrow \bigsqcup_{n\geq 0} d_n \sqsubseteq x$

Examples

- $(\mathbb{N}, \leq)$ $\qquad 0 \leq 1 \leq 2 \leq \cdots \leq n \leq \cdots (n \in \mathbb{N})$

  L does not have lubs of countable chains

- $(\mathbb{N} \cup \{\infty\}, \sqsubseteq)$ $\qquad n \sqsubseteq m$ iff $n \leq m$ $\quad (n, m \in \mathbb{N})$

  $n \sqsubseteq \infty \quad (n \in \mathbb{N})$

$$\left(0 \sqsubseteq 1 \sqsubseteq 2 \quad \cdots \sqsubseteq n \sqsubseteq \cdots \sqsubseteq \infty\right)$$

WARNING Whenever you write $\bigsqcup_{n\geq 0} d_n$ you need make sure that $\{d_n\}_{n\geq 0}$ is a chain; that is, $d_0 \sqsubseteq d_1 \sqsubseteq \cdots \sqsubseteq d_n \sqsubseteq \cdots$

$$\frac{}{\perp \sqsubseteq x}$$

$$\frac{}{x_i \sqsubseteq \bigsqcup_{n\geq 0} x_n} \quad (i \geq 0 \text{ and } \langle x_n \rangle \text{ a chain})$$

$$\frac{\forall n \geq 0 \,.\, x_n \sqsubseteq x}{\bigsqcup_{n\geq 0} x_n \sqsubseteq x} \quad (\langle x_i \rangle \text{ a chain})$$

Domain of partial functions, $X \rightharpoonup Y$

Underlying set: all partial functions, f, with domain of definition $dom(f) \subseteq X$ and taking values in Y.

Partial order:

$$f \sqsubseteq g \quad \text{iff} \quad dom(f) \subseteq dom(g) \text{ and } \forall x \in dom(f).\ f(x) = g(x)$$

$$\text{iff} \quad graph(f) \subseteq graph(g)$$

Lub of chain $f_0 \sqsubseteq f_1 \sqsubseteq f_2 \sqsubseteq \ldots$ is the partial function f with $dom(f) = \bigcup_{n \geq 0} dom(f_n)$ and

$$f(x) = \begin{cases} f_n(x) & \text{if } x \in dom(f_n), \text{ some } n \\ \text{undefined} & \text{otherwise} \end{cases}$$

1 $f_i \sqsubseteq f \ \forall i$

2 $(\forall i\ f_i \sqsubseteq g) \Rightarrow f \sqsubseteq g$

$graph(f) = \bigcup_{n \geq 0} graph(f_n)$.

Example $(\mathcal{P}(X), \subseteq)$ is a domain

- $\perp = \emptyset$
- Given $S_0 \subseteq S_1 \subseteq \cdots \subseteq S_n \subseteq \cdots$

  we have the lub given by $\bigcup_{n \geq 0} S_n$.

Domain of partial functions, $X \rightharpoonup Y$

Underlying set: all partial functions, f, with domain of definition $dom(f) \subseteq X$ and taking values in Y.

Partial order:

$$
\begin{aligned}
f \sqsubseteq g \quad &\text{iff} \quad dom(f) \subseteq dom(g) \text{ and} \\
&\qquad \forall x \in dom(f).\ f(x) = g(x) \\
&\text{iff} \quad graph(f) \subseteq graph(g)
\end{aligned}
$$

Lub of chain $f_0 \sqsubseteq f_1 \sqsubseteq f_2 \sqsubseteq \ldots$ is the partial function f with $dom(f) = \bigcup_{n \geq 0} dom(f_n)$ and

$$
f(x) = \begin{cases} f_n(x) & \text{if } x \in dom(f_n), \text{ some } n \\ \text{undefined} & \text{otherwise} \end{cases}
$$

Least element $\perp$ is the totally undefined partial function.

$d \sqsubseteq d \sqsubseteq d \sqsubseteq \cdots$

Some properties of lubs of chains

Let D be a cpo.

1. For $d \in D$, $\bigsqcup_n d = d$.

2. For every chain $d_0 \sqsubseteq d_1 \sqsubseteq \ldots \sqsubseteq d_n \sqsubseteq \ldots$ in D,

$$\bigsqcup_n d_n = \bigsqcup_n d_{N+n}$$

for all $N \in \mathbb{N}$.

$\bigsqcup (d_0 \sqsubseteq d_1 \sqsubseteq \cdots) = \bigsqcup (d_N \sqsubseteq d_{N+1} \sqsubseteq \cdots)$

3. For every pair of chains $d_0 \sqsubseteq d_1 \sqsubseteq \ldots \sqsubseteq d_n \sqsubseteq \ldots$ and $e_0 \sqsubseteq e_1 \sqsubseteq \ldots \sqsubseteq e_n \sqsubseteq \ldots$ in D,

$$\text{if } d_n \sqsubseteq e_n \text{ for all } n \in \mathbb{N} \text{ then } \bigsqcup_n d_n \ \sqsubseteq \ \bigsqcup_n e_n.$$

$$\begin{array}{ccccccccc}
e_0 & \sqsubseteq & e_1 & \sqsubseteq & e_2 & \sqsubseteq & \cdots & & \bigsqcup_n e_n \\
\rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} \\
d_0 & \sqsubseteq & d_1 & \sqsubseteq & d_2 & \sqsubseteq & \cdots & & \bigsqcup_n d_n
\end{array}$$

$$\dfrac{\dfrac{\forall i \quad \overset{\checkmark}{d_i \sqsubseteq e_i} \qquad \overset{\checkmark}{e_i \sqsubseteq \bigsqcup_n e_n}}{\forall i \quad d_i \sqsubseteq \bigsqcup_n e_n}}{\bigsqcup_n d_n \sqsubseteq \bigsqcup_n e_n}$$

$$\bigsqcup_n d_n = \bigsqcup_n d_{n+1}$$

$$\begin{array}{ccccccc} & & d_1 & \sqsubseteq & d_2 & \sqsubseteq & \cdots \\ & & \sqsupseteq & & \sqsupseteq & & \\ d_0 & \sqsubseteq & d_1 & \sqsubseteq & \cdots & & \end{array}$$

$$\Longrightarrow \bigsqcup_n d_n \sqsubseteq \bigsqcup_n d_{n+1}$$

by previous proposition

✓

We need show

$$\frac{\dfrac{\forall i \quad d_i \sqsubseteq \bigsqcup_n d_n}{\forall i \quad d_{i+1} \sqsubseteq \bigsqcup_n d_n}}{\bigsqcup_n d_{n+1} \sqsubseteq \bigsqcup_n d_n}$$

3. For every pair of chains $d_0 \sqsubseteq d_1 \sqsubseteq \ldots \sqsubseteq d_n \sqsubseteq \ldots$ and $e_0 \sqsubseteq e_1 \sqsubseteq \ldots \sqsubseteq e_n \sqsubseteq \ldots$ in D,

   if $d_n \sqsubseteq e_n$ for all $n \in \mathbb{N}$ then $\bigsqcup_n d_n \ \sqsubseteq\ \bigsqcup_n e_n$.

$$\frac{\forall n \geq 0 \,.\, x_n \sqsubseteq y_n}{\bigsqcup_n x_n \sqsubseteq \bigsqcup_n y_n} \quad (\langle x_n \rangle \text{ and } \langle y_n \rangle \text{ chains})$$

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
d_{n0} & \sqsubseteq & d_{n1} & \sqsubseteq & d_{n2} & \sqsubseteq & \cdots \\
\rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & \\
\vdots & & \vdots & & \vdots & & \\
\rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & \\
d_{10} & \sqsubseteq & d_{11} & \sqsubseteq & d_{12} & \sqsubseteq & \cdots \\
\rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & \\
d_{0,0} & \sqsubseteq & d_{0,1} & \sqsubseteq & d_{0,2} & \sqsubseteq & \cdots
\end{array}$$