$[\![\textbf{while } B \textbf{ do } C]\!]$

$$= \sim\!\sim [\![B]\!] \sim [\![C]\!] \text{ — } \quad ?$$

Operationally, we have

while B do C $\equiv$ if B then (C; while B do C)
else skip

Denotationally, we would like

$[\![$while B do C$]\!] = [\![$if B then (C; while B do C)
else skip$]\!]$

$$[\![\text{while } B \text{ do } C]\!]$$

$$= \lambda s.\ f\left([\![B]\!]s,\ [\![\text{while } B \text{ do } C]\!]([\![C]\!]s),\ s\right)$$

$\boxed{?}$ Can this be taken as a definition?

$\boxed{?}$ Is this compositional?

We learn that $[\![\text{while } B \text{ do } C]\!]$ has the interesting property of being a fixed point.

Def A fixed point of a function f is an element a such that $f(a) = a$.

There is a function for which $[\![\text{while } B \text{ do } C]\!]$ is a fixed point, namely

$$\lambda w.\, \lambda s.\ \text{if}\big([\![B]\!]s,\ w([\![C]\!]s),\ s\big) \stackrel{\text{def}}{=} f_{[\![B]\!],[\![C]\!]}$$

So we can try to define:

$$[\![\text{while } B \text{ do } C]\!] = \underline{fix}\big(f_{[\![B]\!],[\![C]\!]}\big)$$

Is a definition provided we define $\underline{fix}$

COMPOSITIONAL!

Fixed point property of $[\![\mathbf{while}\ B\ \mathbf{do}\ C]\!]$

$$[\![\mathbf{while}\ B\ \mathbf{do}\ C]\!] = f_{[\![B]\!],[\![C]\!]}([\![\mathbf{while}\ B\ \mathbf{do}\ C]\!])$$

where, for each $b : State \to \{true, false\}$ and $c : State \rightharpoonup State$, we define

$$f_{b,c} : (State \rightharpoonup State) \to (State \rightharpoonup State)$$

as

$$f_{b,c} = \lambda w \in (State \rightharpoonup State).\, \lambda s \in State.\, if\bigl(b(s), w(c(s)), s\bigr).$$

- Why does $w = f_{[\![B]\!],[\![C]\!]}(w)$ have a solution?
- What if it has several solutions—which one do we take to be $[\![\mathbf{while}\ B\ \mathbf{do}\ C]\!]$?

$\in (State \to State)$

Approximating $[\![\textbf{while } B \textbf{ do } C]\!]$

$f_{[\![B]\!],[\![C]\!]} : (State \to State) \to (State \to State)$

$\perp \in (State \to State)$ ~ The Totally undefined function

$\perp \sqsubseteq [\![\text{while } B \text{ do } C]\!]$

approximates

Consider $f_{[\![B]\!],[\![C]\!]}(\perp)$

$= \lambda s.\ if([\![B]\!]s, \perp([\![C]\!]s), s)$

$$f_{[\![B]\!],[\![C]\!]}(\bot) = \lambda s.\ \mathrm{if}([\![B]\!]s, \bot, s)$$

$$\bot \sqsubseteq f_{[\![B]\!],[\![C]\!]}(\bot) \sqsubseteq [\![\text{while } B \text{ do } C]\!]$$

Consider

$$f_{[\![B]\!],[\![C]\!]}(f_{[\![B]\!],[\![C]\!]}\ \bot)$$

$$= \lambda s.\ \mathrm{if}([\![B]\!]s, (\lambda s'.\ \mathrm{if}([\![B]\!]s', \bot, s'))([\![C]\!]s), s)$$

$$= \lambda s.\ \mathrm{if}([\![B]\!]s, \mathrm{if}([\![B]\!]([\![C]\!]s), \bot, [\![C]\!]s), s)$$

$$\bot \sqsubseteq f_{[\![B]\!],[\![C]\!]}(\bot) \sqsubseteq f^2_{[\![B]\!],[\![C]\!]}(\bot) \sqsubseteq \cdots$$

$$\sqsubseteq \cdots f^n_{[\![B]\!],[\![C]\!]}(\bot) \sqsubseteq$$

$$\cdots \sqsubseteq [\![\text{while } B \text{ do } C]\!]$$

In fact

the limit

$$[\![\text{while } B \text{ do } C]\!] = \bigcup_n f^n_{[\![B]\!],[\![C]\!]}(\bot)$$

$$= \underline{\text{fix}}\,(f_{[\![B]\!],[\![C]\!]})$$

Approximating $[\![\mathbf{while}\ B\ \mathbf{do}\ C]\!]$

$$
\begin{array}{rl}
& {f_{[\![B]\!],[\![C]\!]}}^{n}(\perp) \\
= & \lambda s \in State. \\
& \left\{ \begin{array}{ll}
[\![C]\!]^{k}(s) & \text{if } \exists\, 0 \le k < n.\, [\![B]\!]([\![C]\!]^{k}(s)) = false \\
 & \text{and } \forall\, 0 \le i < k.\, [\![B]\!]([\![C]\!]^{i}(s)) = true \\
 & \\
\uparrow & \text{if } \forall\, 0 \le i < n.\, [\![B]\!]([\![C]\!]^{i}(s)) = true
\end{array} \right.
\end{array}
$$

$\mathrm{graph}(w) = \{ (x, wx) \mid wx \text{ is defined} \}$

$$D \stackrel{\text{def}}{=} (State \rightharpoonup State)$$

approximation

- **Partial order $\sqsubseteq$ on D:**

$w \sqsubseteq w'$ iff for all $s \in State$, if w is defined at s then so is w' and moreover $w(s) = w'(s)$.

iff the graph of w is included in the graph of w'.

- **Least element $\perp \in D$ w.r.t. $\sqsubseteq$:**

$\perp$ = totally undefined partial function

= partial function with empty graph

(satisfies $\perp \sqsubseteq w$, for all $w \in D$).

Topic 2

Least Fixed Points

It's important to generalise.

Examples: (State → State) is a domain

$f_{[\![B]\!],[\![C]\!]} : (State \to State) \to (State \to State)$

is monotone

Thesis

All domains of computation are partial orders with a least element.

All computable functions are mononotic.

functions f s.t. $x \sqsubseteq y \Rightarrow f(x) \sqsubseteq f(y)$

Example: $(\mathcal{P}(S), \subseteq)$ a partially ordered set with least element $\bot = \emptyset$.

Partially ordered sets

A binary relation $\sqsubseteq$ on a set D is a partial order iff it is

reflexive: $\forall d \in D.\ d \sqsubseteq d$

transitive: $\forall d, d', d'' \in D.\ d \sqsubseteq d' \sqsubseteq d'' \Rightarrow d \sqsubseteq d''$

anti-symmetric: $\forall d, d' \in D.\ d \sqsubseteq d' \sqsubseteq d \Rightarrow d = d'$.

Such a pair $(D, \sqsubseteq)$ is called a partially ordered set, or poset.

$$\frac{}{x \sqsubseteq x}$$

$$\frac{x \sqsubseteq y \qquad y \sqsubseteq z}{x \sqsubseteq z}$$

$$\frac{x \sqsubseteq y \qquad y \sqsubseteq x}{x = y}$$

Domain of partial functions, $X \rightharpoonup Y$

Underlying set: all partial functions, f, with domain of definition $dom(f) \subseteq X$ and taking values in Y.

Partial order:

$$
\begin{array}{lll}
f \sqsubseteq g & \text{iff} & dom(f) \subseteq dom(g) \text{ and} \\
 & & \forall x \in dom(f).\ f(x) = g(x) \\
 & \text{iff} & graph(f) \subseteq graph(g)
\end{array}
$$

Monotonicity

- A function $f : D \to E$ between posets is monotone iff

$$\forall d, d' \in D.\ d \sqsubseteq d' \Rightarrow f(d) \sqsubseteq f(d').$$

$$\frac{x \sqsubseteq y}{f(x) \sqsubseteq f(y)} \quad (f \text{ monotone})$$

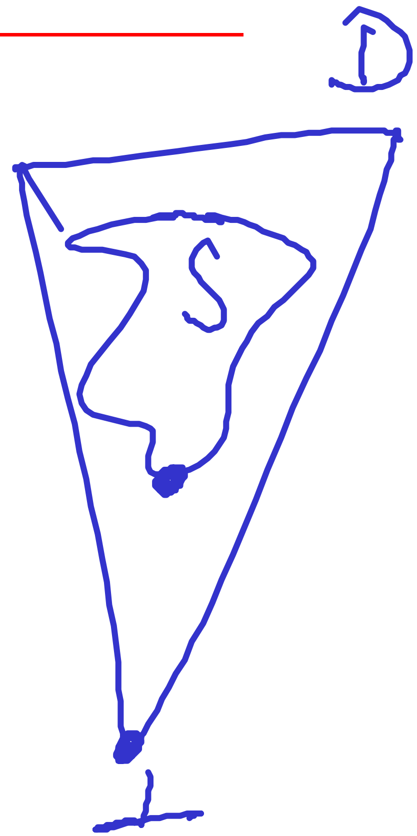
Notation $D = (D, \leq_D) = (D, \sqsubseteq)$

Least Elements

Suppose that D is a poset and that S is a subset of D.

An element $d \in S$ is the *least* element of S if it satisfies

$$\forall x \in S.\, d \sqsubseteq x\ .$$



- Note that because $\sqsubseteq$ is anti-symmetric, S has at most one least element.
- Note also that a poset may not have least element.