M.Phil. Advanced Computer Science
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Introductory logic — Exercise sheet 2

Model theory of first-order logic

Feel free to write me an e-mail if you have questions about, or corrections to, any of the exercises on this sheet. To indicate the difficulty of the problems, I have marked the (hopefully) most accessible exercises with '-' and the difficult ones (which are optional, if you like) with '+'. The exercises that are unmarked fall somewhere in between.

(-) 1. Show that the class of all infinite sets is axiomatisable.

(-) 2. Is the theory of undirected graphs complete? If not, give an example of a first-order sentence which is true in some graphs and false in others

(-) 3. *This exercise asks you to work with elementary definitions of groups; consult e.g. the MathWorld website for the definition.*

   (i) Write down a signature suitable for representing groups as relational structures. Note that this is slightly complicated by the fact that we don't consider function symbols in our signatures.
   (ii) Write down the group axioms as a first-order theory.
   (iii) Is this theory complete?

(-) 4. Let $\mathbf{A}$ be a τ-structure over a domain A. Suppose $\mathbf{B}$ is a τ-structure over a domain $B \subseteq A$ for which it holds that (1) $c^{\mathbf{B}} = c^{\mathbf{A}}$ for all constant symbols c in τ; and (2) $R^{\mathbf{B}} = R^{\mathbf{A}} \cap B^n$ for any n-ary relation symbol R in τ. Then we say that $\mathbf{B}$ is a *substructure* of $\mathbf{A}$ and that $\mathbf{A}$ is an *extension* of $\mathbf{B}$. For example, $(\mathbb{Q}, <^{\mathbb{Q}})$ is a substructure of $(\mathbb{R}, <^{\mathbb{R}})$.

   (i) A first-order theory T is called *universal* if its axioms all have the form $\forall \vec{x} \,.\, \varphi$, where $\vec{x}$ is a (possibly-empty) tuple of variables and φ is quantifier-free. Show that if T is universal then every substructure of a T-model is a T-model.
   (ii) A first-order theory T is called *existential* if its axioms all have the form $\exists \vec{x} \,.\, \varphi$, where $\vec{x}$ is a (possibly-empty) tuple of variables and φ is quantifier-free. Show that if T is existential then every extension of a T-model is a T-model.

5. Suppose $\mathcal{K}$ is a first-order definable class which is can be axiomatised by a theory T; that is to say, $\mathcal{K} = \mathrm{Mod}(T) = \mathrm{Mod}(\varphi)$ for some formula φ which may or may not be in T. Use compactness to show that there is a finite subset $S \subseteq T$ such that $\mathcal{K} = \mathrm{Mod}(S)$.

6. If $\mathcal{K}$ is a class of τ-structures, then the complement of $\mathcal{K}$, written $\bar{\mathcal{K}}$, is the class of all τ-structures not in $\mathcal{K}$.

   (i) Show that a class $\mathcal{K}$ of τ-structures is finitely axiomatisable iff both $\mathcal{K}$ and its complement $\bar{\mathcal{K}}$ are axiomatisable.

   (ii) Conclude that the class of all infinite sets is not finitely axiomatisable (equivalently, not first-order definable).

7. Assume that φ is true in all infinite models of a theory T. Show that there exists a finite number n such that φ is true in *all* models of T that have at least n elements.

(+) 8. Let $G = (V, E^G)$ be a graph, possibly infinite. If $v, w \in V$, then a *path from v to w* is a finite sequence of vertices $v_0, \dots, v_n$ such that $v_0 = v$, $v_n = w$ and for any $i < n$, there is an edge between v_i and v_{i+1}. We say that G is *connected* if there is a path in G between any two vertices in V.

   Now consider the undirected graph $Z = (\mathbb{Z}, E^Z)$ whose vertices are all the integers and where there is an edge between n and m iff $|n - m| = 1$. For example, the number 0 is linked to 1 and -1, 1 is linked to 0 and 2, etc. We can see that Z is a countably infinite line that stretches in both directions. Clearly, Z is connected.

   (i) Use compactness to show that there is an undirected, non-connected graph G which is elementarily equivalent to Z (*hint*: Consider extending the signature by adding a pair of constants, similar to what we did in the proof of the Upward Löwenheim-Skolem theorem).

   (ii) Conclude that the class of connected graphs is not axiomatisable.

   (iii) What can you say about the axiomatisability of the class of non-connected graphs?

(+) 9. *In the lectures, we showed that the theory of dense linear orders without endpoints is $\aleph_0$-categorical. The aim of this question is to show that this is not true in general for arbitrary cardinals.*

   Let $(A, <)$ be an ordered set and write $\leq$ for the reflexive relation $(x < y) \vee (x = y)$. An *upper bound* of a non-empty subset $X \subseteq A$ is an element $b \in A$ with $a \leq b$ for all $a \in A$. An element $u \in A$ is a *least upper bound* (or *l.u.b.*) of X if u is an upper bound of X and if b is also an upper bound of X then $b \leq u$. That is, $\forall b \in X\,.\,(\forall x\,.\,x \leq b) \Rightarrow (u \leq b)$. We say that $(A, <)$ has the *least upper bound property* if every non-empty subset of A has a l.u.b.

   (i) Consider the structure $\mathbf{R} = (\mathbb{R}, <^{\mathbb{R}})$, where $<^{\mathbb{R}}$ is the usual ordering of the reals. Let $\mathbf{C} = (\mathbb{C}, <^{\mathbb{C}})$ be an ordering of the complex numbers where we let

   $$a + i \cdot b <^{\mathbb{C}} x + i \cdot y$$

   if either $(a <^{R} x)$ or $((a = b)$ and $b <^{\mathbb{R}} y)$ (a lexicographical ordering). Show that $\mathbf{R} \equiv \mathbf{C}$.

   (ii) It is well-known that $\mathbf{R}$ has the least upper bound property; show that $\mathbf{C}$ does not.

   (iii) Conclude that $\mathbf{R}$ and $\mathbf{C}$ are not isomorphic.

   (iv) What does this say about the categoricity of dense linear orderings without endpoints?