Lecture 6: functional programming

FreshML

It aimed to provide higher-order structural recursion that automatically respects α-conversion of bound names, without anonymizing binding constructs.

FreshML

Design motivated by simple denotational model in $\mathbf{Nom}$:

nominal sets inductively defined using
$(-) \times (-)$, $[\mathbb{A}](-)$, etc.
$+$
"α-structural" recursion principle

FreshML

Design motivated by simple denotational model in **Nom**:

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"α-structural" recursion principle

How to deal with its freshness side-conditions?

α-Structural recursion

For λ-terms:

Theorem.
Given any $X \in \mathbf{Nom}$ and $\begin{cases} f_1 & \in & \mathbb{A} \to_{\mathrm{fs}} X \\ f_2 & \in & X \times X \to_{\mathrm{fs}} X \\ f_3 & \in & \mathbb{A} \times X \to_{\mathrm{fs}} X \end{cases}$ s.t.

$$(\forall a)\ a \# (f_1, f_2, f_3) \Rightarrow (\forall x)\ a \# f_3(a, x) \qquad \text{(FCB)}$$

$\exists!\ \hat{f} \in \Lambda \to_{\mathrm{fs}} X$ s.t. $\begin{cases} \hat{f}\, a = f_1\, a \\ \hat{f}\,(e_1\, e_2) = f_2(\hat{f}\, e_1, \hat{f}\, e_2) \\ \hat{f}(\lambda a.e) = f_3(a, \hat{f}\, e) \quad \text{if } a \# (f_1, f_2, f_3) \end{cases}$

Can we avoid explicit reasoning about finite support, # and (FCB) when computing 'mod α'?

Want definition/computation to be separate from proving.

FreshML

Design motivated by simple denotational model in $\mathbf{Nom}$:

nominal sets inductively defined using
$(-) \times (-)$, $[\mathbb{A}](-)$, etc.
$+$
"$\boldsymbol{\alpha}$-structural" recursion principle

How to deal with freshness side-conditions?

Pure: type inference (Gabbay-P)
assertion-checking (Pottier)

Impure: dynamically allocated global names
(Shinwell-P)

$$
\begin{array}{rcl}
\hat{f} & = & f_1\, a \\
\hat{f}(e_1\, e_2) & = & f_2(\hat{f}\, e_1, \hat{f}\, e_2) \\
\hat{f}(\lambda a.\, e) & = & f_3(a, \hat{f}\, e) \qquad \text{if } a \# (f_1, f_2, f_2)
\end{array}
$$

$$= \lambda a'.\, e' \qquad\qquad = f_3(a', \hat{f}\, e')$$

Q: how to get rid of this inconvenient proof obligation?

$$\begin{aligned} \hat{f} &= f_1\, a \\ \hat{f}(e_1\, e_2) &= f_2(\hat{f}\, e_1, \hat{f}\, e_2) \\ \hat{f}(\lambda a.\, e) &= \nu a.\, f_3(a, \hat{f}\, e) \quad [\, a \,\#\, (f_1, f_2, f_2)\,] \end{aligned}$$

$= \lambda a'.\, e'$ $\quad\quad$ $= \nu a'.\, f_3(a', \hat{f}\, e')$ ***OK!***

Q: how to get rid of this inconvenient proof obligation?

A: use a local scoping construct $\nu a.\,(-)$ for names

$$\begin{aligned} \hat{f} &= f_1\, a \\ \hat{f}(e_1\, e_2) &= f_2(\hat{f}\, e_1, \hat{f}\, e_2) \\ \hat{f}(\lambda a.\, e) &= \nu a.\, f_3(a, \hat{f}\, e) \quad [\, a \# (f_1, f_2, f_2)\,] \end{aligned}$$

$= \lambda a'.\, e'$ $\quad$ $= \nu a'.\, f_3(a', \hat{f}\, e')$ ***OK!***

Q: how to get rid of this inconvenient proof obligation?

A: use a local scoping construct $\nu a.\, (-)$ for names

which one?!

Dynamic allocation

- Stateful: $\nu a.t$ means "add a fresh name a' to the current state and return $t[a'/a]$".
- Used in Shinwell's Fresh OCaml = OCaml +
  - name types and name-abstraction type former
  - name-abstraction patterns
    —matching involves dynamic allocation of fresh names

  [www.fresh-ocaml.org].

So $\nu a.t$ behaves like ML's

```
let a = ref () in t
```

(using the ML type unit ref as a type of names)

Sample Fresh OCaml code

Syntax uses «_»_ rather than _._

```
(* syntax *)
type t;;
type var = t name;;
type term = Var of var | Lam of «var»term | App of term*term;;

 (* semantics *)
type sem = L of ((unit -> sem) -> sem) | N of neu
and  neu = V of var | A of neu*sem;;

 (* reify : sem -> term *)
let rec reify d =
  match d with L f -> let x = fresh in Lam(«x»(reify(f(function () -> N(V x)))))
             | N n -> reifyn n
and reifyn n =
  match n with V x -> Var x
             | A(n',d') -> App(reifyn n', reify d');;

(* evals : (var * (unit -> sem))list -> term -> sem *)
let rec evals env t  =
  match t with Var x -> (match env with [] -> N(V x)
                                      | (x',v)::env -> if x=x' then v() else evals env (Var x))
             | Lam(«x»t) -> L(function v -> evals ((x,v)::env) t)
             | App(t1,t2) -> (match evals env t1 with L f -> f(function () -> evals env t2)
                                                     | N n -> N(A(n,evals env t2)));;

(* eval : term -> sem *)
let rec eval t = evals [] t;;

(* norm : lam -> lam *)
let norm t = reify(eval t);;
```

Dynamic allocation

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See [www.fresh-ocaml.org].

Dynamic allocation

- Stateful: $\nu a.t$ means "add a fresh name a' to the current state and return $t[a'/a]$".

Statefulness disrupts familiar mathematical properties of pure datatypes. So we will try to reject it in favour of...

Odersky's $\nu a.(-)$

[M. Odersky, *A Functional Theory of Local Names*, POPL'94]

- Unfamiliar—apparently not used in practice (so far).
- Pure equational calculus, in which local scopes 'intrude' rather than extrude (as per dynamic allocation):

$$\begin{array}{rcll} \nu a.(\lambda x.t) & \approx & \lambda x.(\nu a.t) & [a \neq x] \\ \nu a.(t,t') & \approx & (\nu a.t, \nu a.t') & \end{array}$$

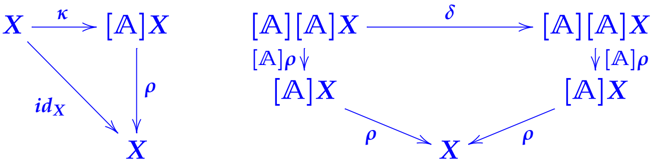
- New: a straightforward semantics using nominal sets equipped with a 'name-restriction operation'...

Name-restriction

A name-restriction operation on a nominal set X is a morphism $(-)\backslash(-) \in \mathbf{Nom}(\mathbb{A} \times X, X)$ satisfying

- $a \# a\backslash x$
- $a \# x \;\Rightarrow\; a\backslash x = x$
- $a\backslash(b\backslash x) = b\backslash(a\backslash x)$

Equivalently, a morphism $\rho : [\mathbb{A}]X \to X$ making



commute, where $\kappa\, x = \langle a\rangle x$ for some (or indeed any) $a \# x$; and where $\delta(\langle a\rangle\langle a'\rangle x) = \langle a'\rangle\langle a\rangle x$.

Given any $X \in \mathbf{Nom}$ and $\begin{cases} f_1 & \in & \mathbb{A} \to_{\mathrm{fs}} X \\ f_2 & \in & X \times X \to_{\mathrm{fs}} X \\ f_3 & \in & \mathbb{A} \times X \to_{\mathrm{fs}} X \end{cases}$ s.t.

$$(\forall a)\ a \# (f_1, f_2, f_3) \Rightarrow (\forall x)\ a \# f_3(a, x) \qquad \text{(FCB)}$$

$\exists!\ \hat{f} \in \Lambda \to_{\mathrm{fs}} X$ .t. $\begin{cases} \hat{f}\, a = f_1\, a \\ \hat{f}\,(e_1\, e_2) = f_2(\hat{f}\, e_1, \hat{f}\, e_2) \\ \hat{f}(\lambda a.e) = f_3(a, \hat{f}\, e) \quad \text{if } a \# (f_1, f_2, f_3) \end{cases}$

If X has a name restriction operation $(-)\backslash(-)$, we can trivially satisfy (FCB) by using $a \backslash f_3(a, x)$ in place of $f_3(a, x)$.

Given any $X \in \mathbf{Nom}$ and $\begin{cases} f_1 & \in & \mathbb{A} \to_{\mathrm{fs}} X \\ f_2 & \in & X \times X \to_{\mathrm{fs}} X \\ f_3 & \in & \mathbb{A} \times X \to_{\mathrm{fs}} X \end{cases}$

and a restriction operation $(-)\backslash(-)$ on X,

$\exists!\ \hat{f} \in \Lambda \to_{\mathrm{fs}} X$ .t. $\begin{cases} \hat{f}\, a = f_1\, a \\ \hat{f}\,(e_1\, e_2) = f_2(\hat{f}\, e_1, \hat{f}\, e_2) \\ \hat{f}(\lambda a.e) = a \backslash f_3(a, \hat{f}\, e) \end{cases}$

Is requiring X to carry a name-restriction operation much of a hindrance for applications?

Not much. . .

Examples of name-restriction

- For $\mathbb{N}$: $$a \backslash n \triangleq n$$

Examples of name-restriction

- For $\mathbb{N}$: $$a \backslash n \triangleq n$$
- For $\mathbb{A}' \triangleq \mathbb{A} \uplus \{\textbf{anon}\}$:

$$\begin{aligned} a \backslash a &\triangleq \textbf{anon} \\ a \backslash a' &\triangleq a' \quad \text{if } a' \neq a \\ a \backslash \textbf{anon} &\triangleq \textbf{anon} \end{aligned}$$

Examples of name-restriction

- For $\mathbb{N}$:
$$a \backslash n \triangleq n$$

- For $\mathbb{A}' \triangleq \mathbb{A} \uplus \{\mathbf{anon}\}$:
$$a \backslash t \triangleq t[\mathbf{anon}/a]$$

- For $\Lambda' \triangleq \{t ::= \mathtt{V}\, a \mid \mathtt{A}(t\,,t) \mid \mathtt{L}(a\,.\,t) \mid \mathbf{anon}\}/{=_\alpha}$:
$$a \backslash [t]_\alpha \triangleq [t[\mathbf{anon}/a]]_\alpha$$

eg. $a \backslash (\lambda b.\, ab) = \lambda b.\, \mathrm{anon}\, b$
$a \backslash (\lambda b.\lambda a.\, ab) = \lambda b.\lambda a.\, ab$
etc.

Examples of name-restriction

- For $\mathbb{N}$: $$a \backslash n \triangleq n$$

- For $\mathbb{A}' \triangleq \mathbb{A} \uplus \{\mathbf{anon}\}$:
$$a \backslash t \triangleq t[\mathbf{anon}/a]$$

- For $\Lambda' \triangleq \{t ::= \mathtt{V}\, a \mid \mathtt{A}(t\,,t) \mid \mathtt{L}(a\,.\,t) \mid \mathbf{anon}\}/{=_\alpha}$:
$$a \backslash [t]_\alpha \triangleq [t[\mathbf{anon}/a]]_\alpha$$

- Nominal sets with name-restriction are closed under products, coproducts, name-abstraction and exponentiation by a nominal set.

$\lambda\alpha\nu$-Calculus

[AMP, *Structural Recursion with Locally Scoped Names*, JFP 21 (2011) 235–286]

is standard simply-typed λ-calculus with booleans and products, extended with:

- type of names, $\mathtt{Name}$, with terms for
  - names, $a : \mathtt{Name}$ $(a \in \mathbb{A})$
  - equality test, $\_ = \_ : \mathtt{Name} \to \mathtt{Name} \to \mathtt{Bool}$
  - name-swapping, $\dfrac{t : T}{(a \wr a')t : T}$
  - locally scoped names $\dfrac{t : T}{\nu a.t : T}$ (binds a)

  with Odersky-style computation rules, e.g.

$$\nu a.\lambda x.t = \lambda x.\nu a.t$$

$\lambda\alpha\nu$-Calculus

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is standard simply-typed λ-calculus with booleans and products, extended with:

- type of names, $\mathtt{Name}$
- name-abstraction types, $\mathtt{Name}\,.\,T$, with terms for
  - name-abstraction, $\dfrac{t : T}{\alpha a.\, t : \mathtt{Name}\,.\,T}$ (binds a)
  - unbinding, $\dfrac{t : \mathtt{Name}\,.\,T \qquad t' : T'}{\mathtt{let}\; a\,.\,x = t\;\mathtt{in}\; t' : T'}$ (binds a & x in t')

  with computation rule that uses local scoping

$$\mathtt{let}\; a\,.\,x = \alpha a.\, t\;\mathtt{in}\; t' = \nu a.\,(t'[t/x])$$

$\lambda\alpha\nu$-Calculus

Denotational semantics. $\lambda\alpha\nu$-calculus has a straightforward interpretation in **Nom** that is sound for the computation rules—types denote nominal sets equipped with a name-restriction operation:

$$
\begin{array}{rcl}
[\![\mathtt{Bool}]\!] & = & \{\mathbf{true}, \mathbf{false}\} \\
[\![\mathtt{Name}]\!] & = & \mathbb{A} \uplus \{\mathbf{anon}\} \\
[\![T \times T']\!] & = & [\![T]\!] \times [\![T']\!] \\
[\![T \to T']\!] & = & [\![T]\!] \to_{\mathbf{fs}} [\![T]\!] \\
[\![\mathtt{Name}\,.\,T]\!] & = & [\mathbb{A}][\![T]\!]
\end{array}
$$

$[\![\nu a.\, a]\!]$ (→ **anon**)

$\lambda\alpha\nu$-calculus as a FP language

To do: revisit FreshML using Odersky-style local names rather than dynamic allocation

```
names Var : Set

data Term : Set where                        --(possibly open) λ-terms mod α
  V : Var -> Term                            --variable
  A : (Term × Term)-> Term                   --application term
  L : (Var . Term) -> Term                   --λ-abstraction

_/_ : Term -> Var -> Term -> Term            --capture-avoiding substitution
(t / x)(V x′) = if x = x′ then t else V x′
(t / x)(A(t′ , t″)) = A((t / x )t′ , (t / x )t″)
(t / x)(L(x′ . t′)) = L(x′ . (t / x)t′)
```

‘Nominal Agda’ (???)

```
names Var : Set

data Term : Set where                           --(possibly open) λ-terms mod α
  V : Var -> Term                               --variable
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data _==_ (t : Term) : Term -> Set where        --intensional equality
  Refl : t == t                                 --is term equality mod α

eg : (x x′ : Var) ->
  ((V x) / x′)(L(x . V x′)) == L(x′ . V x)      --(λx.x′)[x/x′] = λx′.x
eg x x′ = {! !}
```

Dependent types

- Can the $\lambda\alpha\nu$-calculus be extended from simple to dependent types?
  At the moment I do not see how to do this, because...

$$\frac{\Gamma, a : \texttt{Name} \vdash e : T \qquad a \notin fn(T)}{\Gamma \vdash \nu a.\, e : T}$$

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$$\nu a.\,(e_1\,, e_2) \stackrel{?}{=} (\nu a.\, e_1\,, \nu a.\, e_2)$$

$e_1 : T_1$

$e_2 : T_2[e_1]$

$$\frac{\Gamma, a : \mathtt{Name} \vdash e : T \qquad a \notin fn(T)}{\Gamma \vdash \nu a.\, e : T}$$

$$\nu a.\,(e_1\,,e_2) \overset{?}{=} (\nu a.\, e_1\,, \nu a.\, e_2)$$

$e_1 : T_1$

$e_2 : T_2[e_1]$

$\nu a.\,(e_1\,,e_2) : (x : T_1) \times T_2[x]$
if $a \notin fn(T_1, T_2)$

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$$\nu a.\, (e_1 \,, e_2) \stackrel{?}{=} (\nu a.\, e_1 \,, \nu a.\, e_2)$$

$e_1 : T_1$

$\nu a.\, e_1 : T_1$

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$e_1 : T_1$

$\nu a.\, e_1 : T_1$

$e_2 : T_2[e_1]$

$\nu a.\, e_2 : T_2[\nu a.\, e_1]$???

$\nu a.\,(e_1\,, e_2) : (x : T_1) \times T_2[x]$
if $a \notin fn(T_1, T_2)$

Dependent types

- Can the $\lambda\alpha\nu$-calculus be extended from simple to dependent types?
  At the moment I do not see how to do this, because...
- Instead, is there a useful/expressive form of indexed structural induction mod α using dynamically allocated local names?

  (Recent work of Cheney on DNTT is interesting, but probably not sufficiently expressive.)