Lecture 5: nominal ADTs

Lecture 5 1/12

Algebraic Datatypes

Nominal sets support a syntax-free notion of α -equivalence.

Q: What can we do with it?

Lecture 5 2/12

Nominal Algebraic Datatypes

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Q: What can we do with it?

A: Inductively defined data types in **Nom** that combine the usual operations

- ▶ disjoint union $X_1 + \cdots + X_n$ (for data with different constructor forms)
- ► cartesian product $X_1 \times \cdots \times X_n$ (for constructor of several arguments)

with

▶ name-abstraction [A]X (for constructors that involve binding).

Lecture 5 2/12

Nominal Algebraic Datatypes

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categorically defined as initial algebra for functors

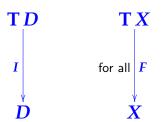
with

▶ name-abstraction [A]X (for constructors that involve binding).

Lecture 5 2/12

Initial algebras

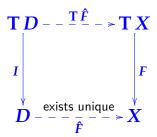
▶ [A](-) has excellent exactness properties. It can be combined with \times , + (and $X \rightarrow_{fs} (-)$) to give functors $T : Nom \rightarrow Nom$ that have initial algebras $I : TD \rightarrow D$



Lecture 5 3/12

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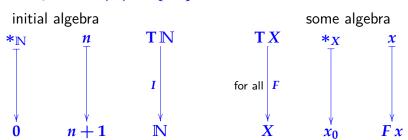


Lecture 5 3/12

E.g. N as an initial algebra

for
$$T(-) = 1 + (-) : Set \rightarrow Set$$
.

Concretely, take $T(X) = \{*_X\} \cup X$ for some $*_X \notin X$.



Lecture 5 4/12

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for
$$T(-)=1+(-):\mathbf{Set}\to\mathbf{Set}.$$

Concretely, take $T(X)=\{*_X\}\cup X$ for some $*_X\notin X$.

initial algebra some algebra

*\mathbb{N} \quad \begin{pmatrix} n & \text{T}\mathbb{N} - \frac{\text{T}\hat{F}}{-\text{F}} - \text{T}\text{X} & *_X & \text{X} \\ \end{pmatrix} & \text{X} & \text{X} & \text{X} \\ \end{pmatrix} & \text{Y} & \text{Y} \\ \end{pmatrix} & \text{Y} \\ \e

Lecture 5 4/12

Initial algebras

- ▶ [A](-) has excellent exactness properties. It can be combined with \times , + (and $X \rightarrow_{fs} (-)$) to give functors $T : Nom \rightarrow Nom$ that have initial algebras $I : TD \rightarrow D$
- For a wide class of such functors (nominal algebraic functors) the initial algebra D coincides with ASTs/α-equivalence.

E.g. Λ is the initial algebra for

$$T(-) \triangleq \mathbb{A} + (-\times -) + [\mathbb{A}](-)$$

Lecture 5 5/12

Example: λ -calculus

name-sort Var for variables, data-sort Term for terms, and operations

 $V: Var \rightarrow Term$

A: Term, Term → Term

 $L: Var. Term \rightarrow Term$

Lecture 5 6/12

```
► Sorts S ::= N name-sort (here just one, for simplicity)

D data-sorts

1 unit

S,S pairs

N.S name-binding
```

► Typed operations op : S → D

Signature Σ is specified by the stuff in red.

Lecture 5 6/12

Example: λ -calculus

name-sort Var for variables, data-sort Term for terms, and operations

 $V: Var \rightarrow Term$

```
A: Term, Term \rightarrow Term

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More examples of operations involving binders:

let x = t in t' \longrightarrow Let: Term, (Var. Term) \rightarrow Term

let ecf x = t in t' \rightarrow Let ecf x = t Let ecf x = t in t' \rightarrow Let ecf x = t L
```

Lecture 5

Example: π -calculus

name-sort Chan for channel names, data-sorts Proc, Pre and Sum for processes, prefixed processes and summations, and operations

```
S: Sum \rightarrow Proc
 Comp: Proc, Proc \rightarrow Proc
    Nu: Chan. Proc \rightarrow Proc
      !: Proc \rightarrow Proc
      P: Pre \rightarrow Sum
      0:1 \rightarrow Sum
 Plus: Sum, Sum \rightarrow Sum
   Out: Chan, Chan, Proc \rightarrow Pre
     In: Chan, (Chan. Proc) \rightarrow Pre
   Tau: Proc \rightarrow Pre
Match: Chan, Chan, Pre \rightarrow Pre
```

Lecture 5 6/12

Closely related notions:

- binding signatures of Fiore, Plotkin & Turi (LICS 1999)
- nominal algebras of Honsell, Miculan & Scagnetto (ICALP 2001)

N.B. all these notions of signature restrict attention to iterated, but *unary* name-binding—there are other kinds of lexically scoped binder (e.g. see Pottier's $C\alpha$ ml language, or Urban's Nominal 2 package for Isabell/HOL.)

Lecture 5 6/12

$\Sigma(S)$ = raw terms over Σ of sort S

$$\frac{a \in \mathbb{A}}{a \in \Sigma(\mathbb{N})} \quad \frac{t \in \Sigma(\mathbb{S}) \quad \text{op: } \mathbb{S} \to \mathbb{D}}{\text{op} \ t \in \Sigma(\mathbb{D})} \quad \frac{() \in \Sigma(\mathbb{1})}{() \in \Sigma(\mathbb{1})}$$

$$\frac{t_1 \in \Sigma(\mathbb{S}_1) \quad t_2 \in \Sigma(\mathbb{S}_2)}{t_1, t_2 \in \Sigma(\mathbb{S}_1, \mathbb{S}_2)} \quad \frac{a \in \mathbb{A} \quad t \in \Sigma(\mathbb{S})}{a \cdot t \in \Sigma(\mathbb{N} \cdot \mathbb{S})}$$

Each $\Sigma(S)$ is a nominal set once equipped with the obvious **Perm** A-action—any finite set of atoms containing all those occurring in t supports $t \in \Sigma(S)$.

Lecture 5 7/1:

Alpha-equivalence $=_{\alpha} \subset \Sigma(S) \times \Sigma(S)$

$$\frac{a \in \mathbb{A}}{a =_{\alpha} a} \qquad \frac{t =_{\alpha} t'}{\operatorname{op} t =_{\alpha} \operatorname{op} t'} \qquad \overline{() =_{\alpha} ()}$$

$$\frac{t_{1} =_{\alpha} t'_{1} \qquad t_{2} =_{\alpha} t'_{2}}{t_{1}, t_{2} =_{\alpha} t'_{1}, t'_{2}}$$

$$\underline{(a_{1} a) \cdot t_{1} =_{\alpha} (a_{2} a) \cdot t_{2}} \qquad a \# (a_{1}, t_{1}, a_{2}, t_{2})$$

$$a_{1} \cdot t_{1} =_{\alpha} a_{2} \cdot t_{2}$$

Lecture 5 8/12

Alpha-equivalence $=_{\alpha} \subseteq \Sigma(S) \times \Sigma(S)$

Fact: $=_{\alpha}$ is equivariant $(t_1 =_{\alpha} t_2 \Rightarrow \pi \cdot t_1 =_{\alpha} \pi \cdot t_2)$ and each quotient

$$\Sigma_{\alpha}(S) \triangleq \{[t]_{\alpha} \mid t \in \Sigma(S)\}$$

is a nominal set with

$$\pi \cdot [t]_{\alpha} = [\pi \cdot t]_{\alpha}$$
 $supp [t]_{\alpha} = fn t$
where
 $fn(a.t) = fn t - \{a\}$
 $fn(t_1, t_2) = fn t_1 \cup fn t_2$
etc.

Lecture 5 8/12

Theorem. Given a nominal algebraic signature Σ (for simplicity, assume Σ has a single data-sort D as well as a single name-sort N) $\Sigma_{\alpha}(D)$ is an initial algebra for the

associated functor $T_{\Sigma}: Nom \rightarrow Nom$.

(NSB p139.)

9/12 9/12

Theorem. Given a nominal algebraic signature Σ (for simplicity, assume Σ has a single data-sort D as well as a single name-sort N)

 $\Sigma_{\alpha}(D)$ is an initial algebra for the associated functor $T_{\Sigma}: Nom \rightarrow Nom$.

$$T_{\Sigma}(-) = \llbracket S_1 \rrbracket (-) + \cdots + \llbracket S_n \rrbracket (-)$$
where Σ has operations $\operatorname{op}_i : S_i \to D \ (i = 1..n)$
and $\llbracket S \rrbracket (-) : \operatorname{\mathbf{Nom}} \to \operatorname{\mathbf{Nom}}$ is defined by:
$$\llbracket \mathbb{N} \rrbracket (-) = \mathbb{A}$$

$$\llbracket \mathbb{D} \rrbracket (-) = (-)$$

$$\llbracket 1 \rrbracket (-) = 1$$

$$\llbracket S_1, S_2 \rrbracket (-) = \llbracket S_1 \rrbracket (-) \times \llbracket S_2 \rrbracket (-)$$

$$\llbracket \mathbb{N} . S \rrbracket (-) = \llbracket \mathbb{A} \rrbracket (\llbracket S \rrbracket (-))$$

Lecture 5 9/12

Theorem. Given a nominal algebraic signature Σ (for simplicity, assume Σ has a single data-sort D as well as a single name-sort N)

 $\Sigma_{\alpha}(D)$ is an initial algebra for the associated functor $T_{\Sigma}: Nom \rightarrow Nom$.

E.g. for the λ -calculus signature with operations

 $V: Var \rightarrow Term$

A: Term, Term \rightarrow Term

 $L: Var. Term \rightarrow Term$

we have

$$T_{\Sigma}(-) = \mathbb{A} + (-\times -) + [\mathbb{A}](-)$$

Lecture 5 9/12

Theorem. Given a nominal algebraic signature Σ (for simplicity, assume Σ has a single data-sort D as well as a single name-sort N)

 $\Sigma_{\alpha}(D)$ is an initial algebra for the associated enriched functor $T_{\Sigma}: Nom \rightarrow Nom$.

 T_{Σ} not only acts on equivariant (=emptily supported) functions, but also on finitely supported functions:

$$\begin{array}{ccc} (X \rightarrow_{\mathsf{fs}} Y) & \to & (\mathsf{T}_{\Sigma} \, X \rightarrow_{\mathsf{fs}} \mathsf{T}_{\Sigma} \, Y) \\ F & \mapsto & \mathsf{T}_{\Sigma} \, F \end{array}$$

E.g. the enriched functor $[\mathbb{A}](-): \mathbf{Nom} \to \mathbf{Nom}$ sends $f \in X \to_{\mathrm{fs}} Y$ to $[\mathbb{A}]f \in [\mathbb{A}]X \to_{\mathrm{fs}} [\mathbb{A}]Y$ where

$$[A] f(\langle a \rangle x) = \langle a \rangle (f x)$$
 if $a \# f$

Lecture 5 9/12

For λ -terms:

Theorem. Given any
$$X \in \mathbf{Nom}$$
 and
$$\begin{cases} f_1 \in \mathbb{A} \to_{\mathrm{fs}} X \\ f_2 \in X \times X \to_{\mathrm{fs}} X \\ f_3 \in [\mathbb{A}]X \to_{\mathrm{fs}} X \end{cases}$$
 $\exists ! \ \hat{f} \in \Lambda \to_{\mathrm{fs}} X$ s.t.
$$\begin{cases} \hat{f} \ a = f_1 \ a \\ \hat{f} \ (e_1 \ e_2) = f_2 (\hat{f} \ e_1, \hat{f} \ e_2) \\ \hat{f} \ (\lambda a.e) = f_3 (a, \hat{f} \ e) \end{cases}$$
 if $a \# (f_1, f_2, f_3)$

10/12

For λ -terms:

Theorem. Given any
$$X \in \mathbf{Nom}$$
 and
$$\begin{cases} f_1 \in \mathbb{A} \to_{\mathrm{fs}} X \\ f_2 \in X \times X \to_{\mathrm{fs}} X \\ f_3 \in \mathbb{A} \times X \to_{\mathrm{fs}} X \end{cases} \text{ s.t.} \\ f_3 \in \mathbb{A} \times X \to_{\mathrm{fs}} X \end{cases}$$
 (\(\forall a\) $a \# (f_1, f_2, f_3) \Rightarrow (\forall x) a \# f_3(a, x)$ (FCB)
$$\exists ! \ \hat{f} \in \Lambda \to_{\mathrm{fs}} X \begin{cases} \hat{f} a = f_1 a \\ \hat{f} (e_1 e_2) = f_2(\hat{f} e_1, \hat{f} e_2) \\ \hat{f} (\lambda a.e) = f_3(a, \hat{f} e) \text{ if } a \# (f_1, f_2, f_3) \end{cases}$$

E.g. capture-avoiding substitution $(-)[e'/a']: \Lambda \to \Lambda$ is the \hat{f} for

$$f_1 a \triangleq \text{if } a = a' \text{ then } e' \text{ else } a$$
 $f_2(e_1, e_2) \triangleq e_1 e_2$
 $f_3(a, e) \triangleq \lambda a.e$

for which (FCB) holds, since $a \# \lambda a.e$

For λ -terms:

Theorem. Given any
$$X \in \mathbf{Nom}$$
 and
$$\begin{cases} f_1 \in \mathbb{A} \to_{\mathrm{fs}} X \\ f_2 \in X \times X \to_{\mathrm{fs}} X \\ f_3 \in \mathbb{A} \times X \to_{\mathrm{fs}} X \end{cases} \text{ s.t.} \\ f_3 \in \mathbb{A} \times X \to_{\mathrm{fs}} X \end{cases}$$

$$(\forall a) \ a \# (f_1, f_2, f_3) \Rightarrow (\forall x) \ a \# f_3(a, x) \qquad (\mathsf{FCB})$$

$$\exists ! \ \hat{f} \in \Lambda \to_{\mathrm{fs}} X \\ \mathsf{s.t.} \begin{cases} \hat{f} \ a = f_1 \ a \\ \hat{f} \ (e_1 \ e_2) = f_2(\hat{f} \ e_1, \hat{f} \ e_2) \\ \hat{f} \ (\lambda a.e) = f_3(a, \hat{f} \ e) \quad \text{if } a \# (f_1, f_2, f_3) \end{cases}$$

E.g. size function $\Lambda \to \mathbb{N}$ is the \hat{f} for

$$\begin{array}{rcl}
f_1 a & \triangleq & 0 \\
f_2(n_1, n_2) & \triangleq & n_1 + n_2 \\
f_3(a, n) & \triangleq & n + 1
\end{array}$$

for which (FCB) holds, since a # (n + 1)

Lecture 5 11/12

For λ -terms:

Theorem.
$$\begin{cases} f_1 \in \mathbb{A} \rightarrow_{\mathrm{fs}} X \\ f_2 \in X \times X \rightarrow_{\mathrm{fs}} X \\ f_3 \in \mathbb{A} \times X \rightarrow_{\mathrm{fs}} X \end{cases} \text{ s.t.} \\ f_3 \in \mathbb{A} \times X \rightarrow_{\mathrm{fs}} X \end{cases}$$

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Non-example: trying to list the bound variables of a λ -term

$$\begin{array}{ccc} f_1 a & \triangleq & \mathsf{nil} \\ f_2(\ell_1, \ell_2) & \triangleq & \ell_1 @ \ell_2 \\ f_3(a, \ell) & \triangleq & a :: \ell \end{array}$$

for which (FCB) does not hold, since $a \in supp(a :: \ell)$.

Lecture 5 11/12

For λ -terms:

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 $(\forall a) \ a \# (f_1, f_2, f_3) \Rightarrow (\forall x) \ a \# f_3(a, x)$ (FCB)
$$\exists ! \ \hat{f} \in \Lambda \to_{\mathrm{fs}} X \\ \text{ s.t. } \begin{cases} \hat{f} \ a = f_1 \ a \\ \hat{f} \ (e_1 \ e_2) = f_2(\hat{f} \ e_1, \hat{f} \ e_2) \\ \hat{f} \ (\lambda a.e) = f_3(a, \hat{f} \ e) \text{ if } a \# (f_1, f_2, f_3) \end{cases}$$

Similar results hold for any nominal algebraic signature—see J ACM 53(2006)459–506.

Implemented in Urban & Berghofer's Nominal package for Isabelle/HOL (classical higher-order logic).

Seems to capture informal usage well, but (FCB) can be tricky...

Lecture 5 11/12

Counting bound variables

For each
$$e \in \Lambda$$
, $cbv \ e \triangleq f \ e \ \rho_0 \in \mathbb{N}$ where we want $f \in \Lambda \to_{fs} X$ with $X = (A \to_{fs} \mathbb{N}) \to_{fs} \mathbb{N}$ to satisfy

$$f a \rho = \rho a$$

$$f(e_1 e_2) \rho = (f e_1 \rho) + (f e_2 \rho)$$

$$f(\lambda a.e) \rho = f e(\rho[a \mapsto 1])$$

and where $\rho_0 \in \mathbb{A} \to_{\mathrm{fs}} \mathbb{N}$ is $\lambda(a \in \mathbb{A}) \to 0$.

12/12 12/12

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$$f(\lambda a.e) \rho = f e(\rho[a \mapsto 1])$$

and where $\rho_0 \in \mathbb{A} \to_{\mathrm{fs}} \mathbb{N}$ is $\lambda(a \in \mathbb{A}) \to 0$.

Looks like we should take $f_3(a,x) = \lambda(\rho \in \mathbb{A} \to_{fs} \mathbb{N}) \to x(\rho[a \mapsto 1])$, but this does not satisfy (FCB). Solution: take X to be a certain nominal subset of $(\mathbb{A} \to_{fs} \mathbb{N}) \to_{fs} \mathbb{N}$. (See NSB p145.)

Lecture 5