# Lecture 4: name abstraction

# Alpha-equivalence

Recall from Lecture 1 the equivalence relation $=_\alpha$ on $Tr \triangleq \{t ::= \mathtt{V}\, a \mid \mathtt{A}(t,t) \mid \mathtt{L}(a,t)\}$

$$\frac{a \in \mathbb{A}}{\mathtt{V}\, a =_\alpha \mathtt{V}\, a} \qquad \frac{t_1 =_\alpha t_1' \qquad t_2 =_\alpha t_2'}{\mathtt{A}(t_1,t_2) =_\alpha \mathtt{A}(t_1',t_2')}$$

$$\frac{(a\ b)\cdot t =_\alpha (a'\ b)\cdot t' \qquad b \notin \{a,a'\} \cup \mathbf{var}(t\ t')}{\mathtt{L}(a,t) =_\alpha \mathtt{L}(a',t')}$$

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this is an instance of the nominal sets notion of 'freshness'

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- In $\mathbb{N}$, $a \# n$ always.
- In $\mathbb{A}$, $a \# b$ iff $a \neq b$.
- In $Tr$, $a \# t$ iff $a \notin \mathbf{var}\, t$
- In $\Lambda$, $a \# [t]_\alpha$ iff $a \notin \mathbf{fv}\, t$.
- In $X \times Y$, $a \# (x, y)$ iff $a \# x$ and $a \# y$.
- In $X \to_{\mathrm{fs}} Y$, $a \# f$ can be subtle!
  (and hence ditto for $\mathbf{P}_{\mathrm{fs}} X$)

# Freshness

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$$a \# x \triangleq a \notin supp\, x$$

Note that if $f \in \mathbf{Nom}(X, Y)$, then for all $x \in X$ we have $supp(f\, x) \subseteq supp\, x$ (Lemma 1 from L3) and hence

$$a \# x \Rightarrow a \# f\, x$$

(More generally, if $f \in X \rightarrow_{\mathrm{fs}} Y$ and $x \in X$, then $a \# f$ and $a \# x$ implies $a \# f\, x$.)

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**Fact:** $\#$ is an equivariant relation

$$(\forall \pi \in \mathbf{Perm}\,\mathbb{A})\; a \# x \Rightarrow \pi\, a \# \pi \cdot x$$

Indeed $\pi \cdot (\mathit{supp}\, x) = \mathit{supp}(\pi \cdot x)$

(Exercise)

[Cf. *Equivariance Principle* — NSB p21]

# Name abstraction

Each $X \in \mathbf{Nom}$ yields a nominal set $[\mathbb{A}]X$ of

name-abstractions $\langle a \rangle x$ are $\sim$-equivalence classes of pairs $(a, x) \in \mathbb{A} \times X$, where

$$(a, x) \sim (a', x') \;\Leftrightarrow\; \exists b \# (a, x, a', x') \\ (b\ a) \cdot x = (b\ a') \cdot x'$$

The $\mathbf{Perm}\,\mathbb{A}$-action on $[\mathbb{A}]X$ is well-defined by

$$\pi \cdot \langle a \rangle x = \langle \pi(a) \rangle (\pi \cdot x)$$

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**Proof.** $\left.\begin{array}{ccc} \mathbb{A} \times X & \to & [\mathbb{A}]X \\ (a,x) & \mapsto & \langle a\rangle x \end{array}\right\}$ is equivariant. So by Lemma 1 from L3, $\mathit{supp}\langle a\rangle x \subseteq \mathit{supp}(a,x) = \{a\} \cup \mathit{supp}\, x$.

**Lemma 2.** $supp(\langle a \rangle x) = supp\, x - \{a\}$, so that

$$b \# \langle a \rangle x \;\Leftrightarrow\; b = a \;\vee\; b \# x$$

**Proof.**

$$\boxed{supp\langle a \rangle x \subseteq \{a\} \cup supp\, x}$$

Note that $(a, x) \sim (a, x') \Rightarrow x = x'$.

So if $\pi\, a = a$ and $\pi \cdot \langle a \rangle x = \langle a \rangle x$, then $\langle a \rangle (\pi \cdot x) = \langle a \rangle x$ and hence $\pi \cdot x = x$.

Therefore, if $A$ supports $\langle a \rangle x$, then $A \cup \{a\}$ supports $x$, and hence $supp\, x \subseteq supp\langle a \rangle x \cup \{a\}$.

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So it suffices to show that $a \# \langle a\rangle x$.

Given $a, x$, pick any $a'$ with $a' \# (a, x)$ and hence also $a' \# \langle a\rangle x$.
So $a = (a'\ a) \cdot a' \# (a'\ a) \cdot \langle a\rangle x$ (by equivariance of #).

But since $a' \# (a, x)$, we get $(a', (a'\ a) \cdot x) \sim (a, x)$ (check).
So $(a'\ a) \cdot \langle a\rangle x = \langle a'\rangle((a'\ a) \cdot x) = \langle a\rangle x$.

Therefore $a \# \langle a\rangle x$. □

# Name abstraction

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$$(a, x) \sim (a', x') \Leftrightarrow \exists b \# (a, x, a', x') \\ (b\ a) \cdot x = (b\ a') \cdot x'$$

We get a functor $[\mathbb{A}](-) : \mathbf{Nom} \to \mathbf{Nom}$ sending $f \in \mathbf{Nom}(X, Y)$ to $[\mathbb{A}]f \in \mathbf{Nom}([\mathbb{A}]X, [\mathbb{A}]Y)$ where

$$[\mathbb{A}]f\,(\langle a \rangle x) = \langle a \rangle (f\ x)$$

# Name abstraction

$[\mathbb{A}](-) : \mathbf{Nom} \to \mathbf{Nom}$ is a kind of (affine) function space—it is right adjoint to the functor $(-) * \mathbb{A} : \mathbf{Nom} \to \mathbf{Nom}$ sending $X$ to $X * \mathbb{A} = \{(x, a) \mid a \# x\}$. (Exercise)

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That explains what morphisms *into* $[\mathbb{A}]X$ look like. More important is the following characterization of morphisms *out of* $[\mathbb{A}]X$ in terms of a 'freshness condition for binders' [NSB p69]. . .

**Theorem 2.** If $f \in \mathbf{Nom}(X \times \mathbb{A} \times Y, Z)$ satisfies

$$(\forall x \in X, a \in \mathbb{A}, y \in Y)\ a \mathbin{\#} x \Rightarrow a \mathbin{\#} f(x, a, y) \qquad \text{(FCB)}$$

then there is a unique $\overline{f} \in \mathbf{Nom}(X \times [\mathbb{A}]Y, Z)$ satisfying

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For example, $X = 1 = \{0\}$, $Y = \mathbb{A}$, $Z = \mathbb{A} + 1 = \mathbb{A} \cup \{0\}$ and $f(0, a, a') = \begin{cases} 0 & \text{if } a = a' \\ a' & \text{if } a \neq a' \end{cases}$ satisfies (FCB), so we get $\overline{f}$ as above and hence $i : [\mathbb{A}]\mathbb{A} \to \mathbb{A} + 1$ with

$$i(\langle a \rangle a') = \begin{cases} 0 & \text{if } a = a' \\ a' & \text{if } a \neq a' \end{cases}.$$

It's not hard to see that $i$ is both injective and surjective, hence an isomorphism in $\mathbf{Nom}$. (Exercise)

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**Proof.** Just have to show that if $a, a' \# x$ then

$$(a, y) \sim (a', y') \ \Rightarrow\ f(x, a, y) = f(x, a', y')$$

so that $f(x, -, -)$ induces a function on equivalence classes. (Equivariance of $\overline{f}$ is automatic.)

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**Proof.** Just have to show that if $a, a' \# x$ then

$$(a, y) \sim (a', y') \ \Rightarrow \ f(x, a, y) = f(x, a', y')$$

But if $(a, y) \sim (a', y')$, then we can find $a'' \# (a, y, a', y', x)$ with $(a''\ a) \cdot y = (a''\ a') \cdot y'$. So

$$\begin{aligned} f(x, a, y) &= (a''\ a) \cdot f(x, a, y) && \text{since } a, a'' \# f(x, a, y) \\ &= f(x, a'', (a''\ a) \cdot y) && \text{since } a, a'' \# x \\ &= f(x, a'', (a''\ a') \cdot y') \\ &= \cdots \\ &= f(x, a', y') && \square \end{aligned}$$

# Some properties of $[\mathbb{A}](-)$

$$[\mathbb{A}](X_1 \times \cdots \times X_n) \cong ([\mathbb{A}]X_1) \times \cdots \times ([\mathbb{A}]X_n)$$

$$[\mathbb{A}](X_1 + \cdots + X_n) \cong ([\mathbb{A}]X_1) + \cdots + ([\mathbb{A}]X_n)$$

$S$ discrete $\Rightarrow [\mathbb{A}]S \cong S$

$$[\mathbb{A}](X \to_{\mathrm{fs}} Y) \cong ([\mathbb{A}]X) \to_{\mathrm{fs}} ([\mathbb{A}]Y) \quad (!!)$$