Lecture 3: category of nominal sets

Category of nominal sets, $\mathbf{Nom}$

- objects $X \in \mathbf{Nom}$ are nominal sets
- morphisms $f \in \mathbf{Nom}(X, Y)$ are functions $f \in Y^X$ that are equivariant:

$$(\forall \pi \in \mathbf{Perm}\,\mathbb{A}, x \in X)\ \pi \cdot (f\,x) = f(\pi \cdot x)$$

for all $\pi \in \mathbf{Perm}\,\mathbb{A}$, $x \in X$.

(Clearly these are closed under composition and include identity functions—so we do get a category.)

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(Clearly these are closed under composition and include identity functions—so we do get a category.)

E.g. $\mathbf{Nom}(\mathbb{A}, \mathbb{A}) = \{id_{\mathbb{A}}\}$ because...

Lemma 1. If $f \in \mathbf{Nom}(X, Y)$ and $x \in X$, then

$$supp(f\,x) \subseteq supp\,x$$

For example, if $f \in \mathbf{Nom}(\mathbb{A}, \mathbb{A})$, then for any $a \in \mathbb{A}$

$$\{f\,a\} = supp(f\,a) \subseteq supp\,a = \{a\}$$

so that $f\,a = a$. Hence $f = id_{\mathbb{A}}$.

Lemma 1. If $f \in \mathbf{Nom}(X, Y)$ and $x \in X$, then

$$supp(f\,x) \subseteq supp\,x$$

Proof. Suppose A supports x in X.

So for any $\pi \in \mathbf{Perm}\,\mathbb{A}$, if $(\forall a \in A)\ \pi\,a = a$, then $\pi \cdot x = x$ and hence $\pi \cdot (f\,x) = f(\pi \cdot x) = f\,x$.

Hence A also supports $f\,x$ in Y.

Taking $A = supp\,x$, we get that $supp\,x$ supports $f\,x$, so $supp(f\,x)$ is contained in $supp\,x$. $\square$

Nom is a cartesian closed category

Finite products: $X_1 \times \cdots \times X_n$ is given by cartesian product of sets with **Perm** $\mathbb{A}$-action

$$\pi \cdot (x_1, \ldots, x_n) \triangleq (\pi \cdot x_1, \ldots, \pi \cdot x_n)$$

which satisfies

$$supp(x, \ldots, x_n) = (supp\, x_1) \cup \cdots \cup (supp\, x_n)$$

(Exercise)

$\mathbf{Nom}$ is a cartesian closed category

Exponentials: given $X, Y \in \mathbf{Nom}$, we get a $\mathbf{Perm}\,\mathbb{A}$-action on the set Y^X of functions:

$$\pi \cdot f \triangleq \lambda(x \in X) \to \pi \cdot (f(\pi^{-1} \cdot x))$$

Not every $f \in Y^X$ need have finite support wrt this action: let $X \to_{\mathrm{fs}} Y$ be the subset of ones that do.

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E.g. given $a \in \mathbb{A}$, $K_a \triangleq \lambda(x \in \mathbb{A}) \to a$ is in $\mathbb{A} \to_{\mathrm{fs}} \mathbb{A}$, because it is supported by $\{a\}$, since

$$(\pi \cdot K_a)\, x = \pi \cdot (K_a(\pi^{-1} \cdot x)) = \pi \cdot a = \pi\, a$$

and hence $\pi \cdot K_a = K_{\pi\, a}$.

$\mathbf{Nom}$ is a cartesian closed category

Exponential of $X, Y \in \mathbf{Nom}$ is $X \to_{\mathrm{fs}} Y$ plus equivariant function

$$\begin{aligned} \mathbf{app} &: (X \to_{\mathrm{fs}} Y) \times X \to Y \\ \mathbf{app}\,(f, x) &= f\,x \end{aligned}$$

Given $f \in \mathbf{Nom}(Z \times X, Y)$, the unique $\hat{f} \in \mathbf{Nom}(Z, X \to_{\mathrm{fs}} Y)$ making

$$\begin{tikzcd} Z \times X \arrow[d, "\hat{f} \times id_X"'] \arrow[dr, "f"] & \\ (X \to_{\mathrm{fs}} Y) \times X \arrow[r, "\mathbf{app}"'] & Y \end{tikzcd}$$

commute is given by currying: $\hat{f}\,z = \lambda(x \in X) \to f(z, x)$.

(Exercise)

Nom is a model of Church's higher order logic

[**Nom** is categorically equivalent to a well-known Boolean topos, called the Schanuel topos.]

$\mathbf{Nom}$ is a model of Church's higher order logic

Coproducts are given by disjoint union.

Natural number object: $\mathbb{N} = \{0, 1, 2, \ldots\}$ with trivial $\mathbf{Perm}\,\mathbb{A}$-action: $\pi \cdot n \triangleq n$ (so $supp\, n = \emptyset$).

Nom is a model of Church's higher order logic

Subobject classifier: $\Omega = \{\mathbf{true}, \mathbf{false}\} \cong \mathbf{1} + \mathbf{1}$ with trivial $\mathbf{Perm}\,\mathbb{A}$-action: $\pi \cdot b \triangleq b$ (so $supp\, b = \emptyset$).

Power objects: $X \to_{\mathbf{fs}} \Omega \cong \mathbf{P_{fs}}\, X$, the set of subsets $S \subseteq X$ that are finitely supported w.r.t. the $\mathbf{Perm}\,\mathbb{A}$-action

$$\pi \cdot S \triangleq \{\pi \cdot x \mid x \in S\}$$

Nom $\not\models$ choice

Nom models classical higher-order logic, but not Hilbert's ε-operation, $\varepsilon x.\varphi(x)$ satisfying

$$(\forall x : X)\ \varphi(x) \Rightarrow \varphi(\varepsilon x.\varphi(x))$$

Theorem 1. There is no equivariant function $c : \{S \in \mathbf{P}_{\mathrm{fs}}\, \mathbb{A} \mid S \neq \emptyset\} \to \mathbb{A}$ satisfying $c(S) \in S$ for all non-empty $S \in \mathbf{P}_{\mathrm{fs}}\, \mathbb{A}$.

Proof. Suppose there were such a c. Putting $a \triangleq c\, \mathbb{A}$ and picking some $b \in \mathbb{A} - \{a\}$, we get a contradiction to $a \neq b$:

$$a = c\,\mathbb{A} = c((a\ b) \cdot \mathbb{A}) = (a\ b) \cdot c\,\mathbb{A} = (a\ b) \cdot a = b$$

Nom $\not\models$ choice

Nom models classical higher-order logic, but not Hilbert's ε-operation, $\varepsilon x.\varphi(x)$ satisfying

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In fact **Nom** does not model even very weak forms of choice, such as Dependent Choice.

The nominal set of names

Recall that $\mathbb{A}$ is a nominal set once equipped with the action

$$\pi \cdot a = \pi(a)$$

which satisfies $\mathit{supp}\, a = \{a\}$.

Although $\mathbb{A} \in \mathbf{Set}$ is a countable, $\mathbb{A}$ is not isomorphic to $\mathbb{N}$ in $\mathbf{Nom}$. For any $f \in \mathbb{N} \to_{\mathbf{fs}} \mathbb{A}$ has to satisfy

$$\{f\,n\} = \mathit{supp}(\mathbf{app}(f, n)) \subseteq \mathit{supp}(f, n) = \mathit{supp}\, f \cup \mathit{supp}\, n = \mathit{supp}\, f$$

for all $n \in \mathbb{N}$, and so f cannot be surjective.