MPhil ACS, CST Part III 2012/13
Nominal Sets and their Applications
Exercise Sheet

[* indicates a harder exercise]

Exercise 1. Let Tr, var : $Tr \to \mathrm{P_f}\mathbb{A}$ and $=_\alpha \subseteq Tr \times Tr$ be as in Lecture 1.

(i) Prove by induction on the structure of abstract syntax trees t that the action $(-) \cdot (-)$: $\mathrm{Perm}\,\mathbb{A} \times Tr \to Tr$ defined in Lecture 2 satisfies $\mathrm{var}(\pi \cdot t) = \{\pi\, a \mid a \in \mathrm{var}\, t\}$.

(ii) Show that for any $a, a' \in \mathbb{A}$ and $\pi \in \mathrm{Perm}\,\mathbb{A}$, $\pi \circ (a\ a') = (\pi\, a\ \pi\, a') \circ \pi$ in $\mathrm{Perm}\,\mathbb{A}$.

(iii) Hence prove by induction on the derivation of $t =_\alpha t'$ from the rules inductively defining $=_\alpha$ that if $t =_\alpha t'$, then $\pi \cdot t =_\alpha \pi \cdot t'$ holds for any $\pi \in \mathrm{Perm}\,\mathbb{A}$.

[If you are not confident about proofs by structural induction and rule-based induction, why not try formulating your proofs in Agda, Coq or Isabelle/HOL.]

Exercise 2. Use Exercise 1 to show that if $(a\ \ b) \cdot t =_\alpha (a'\ \ b) \cdot t'$ holds for some $b \in \mathbb{A} - (\{a, a'\} \cup \mathrm{var}(t\, t'))$, then it holds for any such b. Use this to prove that $=_\alpha$ is an equivalence relation.

Exercise 3. The finite set fv t of free variables of $t \in Tr$ is recursively defined by:

$$\begin{aligned} \mathrm{fv}(\mathtt{V}\, a) &= \{a\} \\ \mathrm{fv}(\mathtt{A}(t, t')) &= (\mathrm{fv}\, t) \cup \mathrm{fv}\, t') \\ \mathrm{fv}(\mathtt{L}(a, t)) &= (\mathrm{fv}\, t) - \{a\}. \end{aligned}$$

(i) Prove that for all $\pi \in \mathrm{Perm}\,\mathbb{A}$ and $t \in Tr$, $\mathrm{fv}(\pi \cdot t) = \{\pi\, a \mid a \in \mathrm{fv}\, t\}$.

(ii)* Prove that for all $t \in Tr$, $((\forall a \in \mathrm{fv}\, t)\ \pi\, a = a) \Leftrightarrow \pi \cdot t =_\alpha t$.
[Hint: proceed by induction on the size $|t|$ of abstract syntax trees t, where $|\mathtt{V}\, a| = 0$, $|\mathtt{A}(t, t')| = |t| + |t'| + 1$ and $|\mathtt{L}(a, t)| = |t| + 2$, say. Note that $|(a\ \ a') \cdot t| = |t|$, so that in the induction step for $\mathtt{L}(a, t)$ one can suitably freshen the bound variable, $\mathtt{L}(a, t) =_\alpha \mathtt{L}(a', (a\ a') \cdot t)$, and apply the induction hypothesis to $(a\ a') \cdot t$.]

(iii) Deduce that the smallest support of the α-equivalence class $[t]_\alpha$ in $\Lambda = \{[t]_\alpha \mid t \in Tr\}$ is fv t.

Exercise 4. (i) Show that in the category **Nom** the product of two objects X and Y is given by their cartesian product as sets $X \times Y = \{(x, y) \mid x \in X \wedge y \in Y\}$ with $\mathrm{Perm}\,\mathbb{A}$-action $\pi \cdot (x, y) = (\pi \cdot x, \pi \cdot y)$.

(ii) What is the terminal object 1 in **Nom**?

(iii) Prove that for all $(x, y) \in X \times Y$, $supp(x, y) = supp\, x \cup supp\, y$.

Exercise 5. If $X \in$ **Nom**, $x \in X$ and $A \in \mathrm{P_f}\mathbb{A}$, show that for all $\pi \in \mathrm{Perm}\,\mathbb{A}$ that if A supports x, then $\pi \cdot A \triangleq \{\pi\, a \mid a \in A\}$ supports $\pi \cdot x$. Deduce that $supp(\pi \cdot x) = \pi \cdot (supp\, x)$.

Exercise 6. Show that $f \in \mathbf{Nom}(X, Y)$ is an isomorphism iff the function f is not only equivariant, but also a bijection.

Exercise 7. Continuing Exercise 4, show that **Nom** is a cartesian closed category. To do this, show that the exponential of two nominal sets X and Y is given by the nominal set $X \to_{\mathrm{fs}} Y$ of finitely supported functions defined in Lecture 3.

Exercise 8. Show that the name abstraction functor $[\mathbb{A}](-) : \mathbf{Nom} \to \mathbf{Nom}$ is right adjoint to the functor $(-) * \mathbb{A} : \mathbf{Nom} \to \mathbf{Nom}$ which sends each $X \in \mathbf{Nom}$ to

$$X * \mathbb{A} \triangleq \{(x, a) \in X \times \mathbb{A} \mid a \mathrel{\#} x\}$$

(with Perm $\mathbb{A}$-action inherited from the product $X \times \mathbb{A}$) and each $f \in \mathbf{Nom}(X, Y)$ to $f * \mathbb{A} \in \mathbf{Nom}(X * \mathbb{A}, Y * \mathbb{A})$, given by $(f * \mathbb{A})(x, a) = (f\,x, a)$.

To do this, first show that there is a well-defined equivariant function $(-) \mathbin{@} (-) : ([\mathbb{A}]X) * \mathbb{A} \to X$ satisfying $(\langle a \rangle x) \mathbin{@} b = (a\ b) \cdot x$. This is called *concretion* and is the counit of the adjunction: show that if $f \in \mathbf{Nom}(Y * \mathbb{A}, X)$, then there is a unique morphism $\hat{f} \in \mathbf{Nom}(Y, [\mathbb{A}]X)$ satisfying $f(y, a) = (\hat{f}\,y) \mathbin{@} a$, for all $(y, a) \in Y * \mathbb{A}$.

Exercise 9. Coproducts in **Nom** are given by disjoint union, $X + Y \triangleq \{(0, x) \mid x \in X\} \cup \{(1, y) \mid y \in Y\}$ with Perm $\mathbb{A}$-action given by $\begin{cases} \pi \cdot (0, x) = (0, \pi \cdot x) \\ \pi \cdot (1, y) = (1, \pi \cdot y). \end{cases}$
Show that $[\mathbb{A}](X + Y)$ is isomorphic to $([\mathbb{A}]X) + ([\mathbb{A}]Y)$.

Exercise 10. Show that $[\mathbb{A}]\mathbb{A}$ is isomorphic in the category **Nom** to the coproduct $\mathbb{A} + 1$.

Exercise 11. For any discrete nominal set S (cf. Lecture 2), show that $[\mathbb{A}]S$ is isomorphic to S in **Nom**.

Exercise 12. Show that for any $X, Y \in \mathbf{Nom}$, $[\mathbb{A}](X \times Y)$ is isomorphic to $([\mathbb{A}]X) \times ([\mathbb{A}]Y)$.

Exercise* 13. Show that for any $X, Y \in \mathbf{Nom}$, $[\mathbb{A}](X \to_{\mathrm{fs}} Y)$ is isomorphic to $([\mathbb{A}]X) \to_{\mathrm{fs}} ([\mathbb{A}]Y)$.

Exercise 14. Suppose $\varphi(a)$ and $\varphi'(a)$ are properties of atomic names $a \in \mathbb{A}$ whose extensions $\{a \mid \varphi(a)\}$ and $\{a \mid \varphi'(a)\}$ give finitely supported subsets of $\mathbb{A}$. Writing $(\text{И}a)\ \varphi(a)$ to indicate that $\{a \mid \varphi(a)\}$ is a cofinite set of atoms (cf. Lecture 7), show that this 'freshness quantifier' has the following properties:

(i) $\neg(\text{И}a)\ \varphi(a) \;\Leftrightarrow\; (\text{И}a)\ \neg\varphi(a)$.

(ii) $((\text{И}a)\ \varphi(a) \wedge (\text{И}a)\ \varphi'(a)) \;\Leftrightarrow\; (\text{И}a)\ (\varphi(a) \wedge \varphi'(a))$.

(iii) $((\text{И}a)\ \varphi(a) \vee (\text{И}a)\ \varphi'(a)) \;\Leftrightarrow\; (\text{И}a)\ (\varphi(a) \vee \varphi'(a))$.

(iv) $((\text{И}a)\ \varphi(a) \Rightarrow (\text{И}a)\ \varphi'(a)) \;\Leftrightarrow\; (\text{И}a)\ (\varphi(a) \Rightarrow \varphi'(a))$.

If $X \in \mathbf{Nom}$ and $\varphi(a, x)$ determines a finitely supported subset of $\mathbb{A} \times X$, what in general is the relationship between $(\exists x \in X)(\text{И}a)\ \varphi(a, x)$ and $(\text{И}a)(\exists x \in X)\ \varphi(a, x)$? And between $(\forall x \in X)(\text{И}a)\ \varphi(a, x)$ and $(\text{И}a)(\forall x \in X)\ \varphi(a, x)$?

Exercise 15. Use the α-structural recursion theorem for λ-terms from Lecture 5 to prove the following *α-structural induction principle* for the nominal set Λ of λ-terms modulo α-equivalence: if $P \in \mathrm{P_{fs}}\Lambda$ satisfies

$$
\begin{aligned}
&(\forall a \in \mathbb{A})\ a \in P \\
&\wedge (\forall e_1, e_2 \in \Lambda)\ e_1 \in P \wedge e_2 \in P \Rightarrow e_1\, e_2 \in P \\
&\wedge (\text{И}a)(\forall e \in \Lambda)\ e \in P \Rightarrow \lambda a.\, e \in P
\end{aligned}
$$

then $(\forall e \in \Lambda)\ e \in P$. [Hint: for any nominal set X, $\mathrm{P_{fs}}X$ is isomorphic to $X \to_{\mathrm{fs}} 2$; so we can apply the recursion principle to functions from Λ to 2.]

Exercise 16. Show that a subset S of the nominal set $\mathbb{A}$ is finitely supported iff it is either finite or cofinite (that is, its complement $\mathbb{A} - S$ is finite).

Exercise 17. (i) or each $X \in \mathbf{Nom}$, show that

$$a \backslash S \triangleq \{x \in X \mid (\text{И}a')\,(a\ a') \cdot x \in S\} \qquad (a \in \mathbb{A}, S \in \mathrm{P_{fs}}X)$$

defines a name-restriction operation (Lecture 6) on $\mathrm{P_{fs}}X$.

(ii) When $X = \mathbb{A}$, show that $a \backslash S = S - \{a\}$ if S is finite and $a \backslash S = S \cup \{a\}$ if S is cofinite (cf. Exercise 16).

Exercise* 18. Show that if $(-)\backslash(-) \in \mathbf{Nom}(\mathbb{A} \times Y, Y)$ is a name-restriction operation on $Y \in \mathbf{Nom}$ (Lecture 6), then for any $X \in \mathbf{Nom}$, there is a name restriction operation $(-)\backslash_1(-)$ on $X \to_{\mathrm{fs}} Y$ satisfying

$$a \mathrel{\#} x \Rightarrow (a \backslash_1 f)\, x = a \backslash (f\ x)$$

for all $a \in \mathbb{A}$, $x \in X$ and $f \in X \to_{\mathrm{fs}} Y$.