L11 : Algebraic Path Problems with Applications to Internet Routing
Lecture 11

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Path Weight with functions on arcs?

For graph $G = (V,\ E)$, and arc path $p = (u_0,\ u_1)(u_1,\ u_2)\cdots(u_{k-1},\ u_k)$.

Functions on arcs: two natural ways to do this...

Weight function $w : E \to (S \to S)$. Let $f_j = w(u_{j-1}, u_j)$.

$$
\begin{aligned}
w_a^L(p) &= f_1(f_2(\cdots f_k(a)\cdots)) &= (f_1 \circ f_2 \circ \cdots \circ f_k)(a) \\
w_a^R(p) &= f_k(f_{k-1}(\cdots f_1(a)\cdots)) &= (f_k \circ f_{k-1} \circ \cdots \circ f_1)(a)
\end{aligned}
$$

How can we “make this work” for path problems?

Algebra of Monoid Endomorphisms (See Gondran and Minoux 2008)

Let $(S, \oplus, \overline{0})$ be a commutative monoid.

$(S, \oplus, F \subseteq S \to S, \overline{0}, i, \omega)$ is an algebra of monoid endomorphisms (AME) if

- $\forall f \in F \; \forall b, c \in S : f(b \oplus c) = f(b) \oplus f(c)$
- $\forall f \in F : f(\overline{0}) = \overline{0}$
- $\exists i \in F \; \forall a \in S : i(a) = a$
- $\exists \omega \in F \; \forall a \in S : \omega(a) = \overline{0}$

So why do we need Monoid Endomorphisms??

Monoid Endomorphisms can be viewed as semirings

Suppose $(S, \oplus, F)$ is a monoid of endomorphisms. We can turn it into a semiring

$$(F, \hat{\oplus}, \circ)$$

where $(f \mathbin{\hat{\oplus}} g)(a) = f(a) \oplus g(a)$

Functions are hard to work with....

- All algorithms need to check equality over elements of semiring,
- $f = g$ means $\forall a \in S : f(a) = g(a)$,
- S can be very large, or infinite.

Left and Right AME of Matrices

Given an AME $S = (S, \oplus, F)$, define the (left or right) AME of $n \times n$-matrices over S as

$$\mathbb{M}_n(S) = (\mathbb{M}_n(S), \oplus, \mathbb{F}),$$

where for $\mathbf{A}, \mathbf{B} \in \mathbb{M}_n(S)$ we have

$$(\mathbf{A} \oplus \mathbf{B})(i, j) = \mathbf{A}(i, j) \oplus \mathbf{B}(i, j).$$

Elements of the set $\mathbb{F}$ are $n \times n$ matrices of functions in F. That is, if $\mathbf{A} \in \mathbb{F}$, then $\mathbf{A}(i, j) \in F$.

Left and Right AME of Matrices

We have two natural options for treating **A** as a function in $\mathbb{M}_n(S) \to \mathbb{M}_n(S)$.

Left application

$$(\mathbf{A}(\mathbf{B}))(i,\ j) = \bigoplus_{1 \le q \le n} \mathbf{A}(i,\ q)(\mathbf{B}(q,\ j))$$

Right application

$$((\mathbf{B})\mathbf{A})(i,\ j) = \bigoplus_{1 \le q \le n} \mathbf{A}(q,\ j)(\mathbf{B}(i,\ q))$$

Check distributivity : $\mathbf{A}(\mathbf{B} \oplus \mathbf{B}) = \mathbf{A}(\mathbf{B}) \oplus \mathbf{A}(\mathbf{B})$ and $(\mathbf{B} \oplus \mathbf{B})\mathbf{A} = (\mathbf{B})\mathbf{A} \oplus (\mathbf{B})\mathbf{A}$.

Solving (some) equations. Left version here ...

We will be interested in solving for **L** equations of the form

$$\mathbf{L} = \mathbf{A}(\mathbf{L}) \oplus \mathbf{M}$$

Let

$$\begin{aligned} \mathbf{A}^0(\mathbf{B}) &= \mathbf{B} \\ \mathbf{A}^{k+1}(\mathbf{B}) &= \mathbf{A}(\mathbf{A}^k(\mathbf{B})) \end{aligned}$$

and

$$\begin{aligned} \mathbf{A}^{(k)}(\mathbf{B}) &= \mathbf{A}^0(\mathbf{B}) \oplus \mathbf{A}^1(\mathbf{B}) \oplus \mathbf{A}^2(\mathbf{B}) \oplus \cdots \oplus \mathbf{A}^k(\mathbf{B}) \\ \mathbf{A}^*(\mathbf{B}) &= \mathbf{A}^0(\mathbf{B}) \oplus \mathbf{A}^1(\mathbf{B}) \oplus \mathbf{A}^2(\mathbf{B}) \oplus \cdots \oplus \mathbf{A}^k(\mathbf{B}) \oplus \cdots \end{aligned}$$

Definition (q stability)

If there exists a q such that for all **B**, $\mathbf{A}^{(q)}(\mathbf{B}) = f^{(q+1)}(\mathbf{B})$, then **A** is *$q$-stable*. Therefore, $\mathbf{A}^{(*)}(\mathbf{B}) = \mathbf{A}^{(q)}(\mathbf{B})$.

Key result (again)

Lemma

If $\mathbf{A}$ *is q-stable, then* $\mathbf{L} = \mathbf{A}^{(*)}(\mathbf{B})$ *solves the AME equation*

$$\mathbf{L} = \mathbf{A}(\mathbf{L}) \oplus \mathbf{B}.$$

Proof: Substitute $\mathbf{A}^{(*)}(\mathbf{B})$ for $\mathbf{L}$ to obtain

$$\begin{aligned}
& \mathbf{A}(\mathbf{A}^{(*)}(\mathbf{B})) \oplus \mathbf{B} \\
= \; & \mathbf{A}(\mathbf{A}^{(q)}(\mathbf{B})) \oplus \mathbf{B} \\
= \; & \mathbf{A}(\mathbf{A}^{0}(\mathbf{B}) \oplus \mathbf{A}^{1}(\mathbf{B}) \oplus \mathbf{A}^{2}(\mathbf{B}) \oplus \cdots \oplus \mathbf{A}^{q}(\mathbf{B})) \oplus \mathbf{B} \\
= \; & \mathbf{A}^{1}(\mathbf{B}) \oplus \mathbf{A}^{1}(\mathbf{B}) \oplus \mathbf{A}^{2}(\mathbf{B}) \oplus \cdots \oplus \mathbf{A}^{q+1}(\mathbf{B}) \oplus \mathbf{B} \\
= \; & \mathbf{A}^{0}(\mathbf{B}) \oplus \mathbf{A}^{1}(\mathbf{B}) \oplus \mathbf{A}^{1}(\mathbf{B}) \oplus \mathbf{A}^{2}(\mathbf{B}) \oplus \cdots \oplus \mathbf{A}^{q+1}(\mathbf{B}) \\
= \; & \mathbf{A}^{(q+1)}(\mathbf{B}) \\
= \; & \mathbf{A}^{(q)}(\mathbf{B}) \\
= \; & \mathbf{A}^{(*)}(\mathbf{B})
\end{aligned}$$

Lexicographic product of AMEs

$$(S,\ \oplus_S,\ F)\ \vec{\times}\ (T,\ \oplus_T,\ G) = (S \times T,\ \oplus_S\ \vec{\times}\ \oplus_T,\ F \times G)$$

Theorem 11.1

$$\mathrm{D}(S\ \vec{\times}\ T) \iff \mathrm{D}(S) \wedge \mathrm{D}(T) \wedge (\mathrm{C}(S) \vee \mathrm{K}(T))$$

Where

Property	Definition
D	$\forall a, b, f : f(a \oplus b) = f(a) \oplus f(b)$
C	$\forall a, b, f : f(a) = f(b) \implies a = b$
K	$\forall a, b, f : f(a) = f(b)$

Functional Union of AMEs

$$(S,\ \oplus,\ F) +_{\mathrm{m}} (S,\ \oplus,\ G) = (S,\ \oplus,\ F + G)$$

Fact

$$\mathrm{D}(S +_{\mathrm{m}} T) \iff \mathrm{D}(S) \wedge \mathrm{D}(T)$$

Where

Property	Definition
D	$\forall a, b, f : f(a \oplus b) = f(a) \oplus f(b)$

Left and Right

right

$$\textbf{right}(S, \oplus, F) = (S, \oplus, \{i\})$$

left

$$\textbf{left}(S, \oplus, F) = (S, \oplus, K(S))$$

where $K(S)$ represents all constant functions over S. For $a \in S$, define the function $\kappa_a(b) = a$. Then $K(S) = \{\kappa_a \mid a \in S\}$.

Facts

The following are always true.

$\text{D}(\textbf{right}(S))$
$\text{D}(\textbf{left}(S))$ (assuming $\oplus$ is idempotent)
$\text{C}(\textbf{right}(S))$
$\text{K}(\textbf{left}(S))$

Scoped Product

$$S \Theta T = (S \vec{\times} \textbf{left}(T)) +_{\mathrm{m}} (\textbf{right}(S) \vec{\times} T)$$

Theorem 11.2

$$\mathrm{D}(S \Theta T) \iff \mathrm{D}(S) \wedge \mathrm{D}(T).$$

Proof.

$$\begin{aligned}
& \mathrm{D}(S \Theta T) \\
& \mathrm{D}((S \vec{\times} \textbf{left}(T)) +_{\mathrm{m}} (\textbf{right}(S) \vec{\times} T)) \\
\iff & \mathrm{D}(S \vec{\times} \textbf{left}(T)) \wedge \mathrm{D}(\textbf{right}(S) \vec{\times} T) \\
\iff & \mathrm{D}(S) \wedge \mathrm{D}(\textbf{left}(T)) \wedge (\mathrm{C}(S) \vee \mathrm{K}(\textbf{left}(T))) \\
& \quad \wedge \mathrm{D}(\textbf{right}(S)) \wedge \mathrm{D}(T) \wedge (\mathrm{C}(\textbf{right}(S)) \vee \mathrm{K}(T)) \\
\iff & \mathrm{D}(S) \wedge \mathrm{D}(T)
\end{aligned}$$

□

How do we represent functions?

Definition (transforms (indexed functions))

A set of transforms $(S,\ L,\ \rhd)$ is made up of non-empty sets S and L, and a function

$$\rhd \in L \to (S \to S).$$

We normally write $l \rhd s$ rather than $\rhd(l)(s)$. We can think of $l \in L$ as the index for a function $f_l(s) = l \rhd s$, so $(S,\ L,\ \rhd)$ represents the set of function $F = \{f_l \mid l \in L\}$.

Examples

Example 1: Trivial

Let $(S, \otimes)$ be a semigroup.

$$\text{transform}(S, \oplus) = (S, S, \rhd_{\otimes}),$$

where $a \rhd_{\otimes} b = a \otimes b$

Example 2: Restriction

For $T \subset S$,

$$\text{Restrict}(T, (S, \oplus)) = (S, T, \rhd_{\otimes}),$$

where $a \rhd_{\otimes} b = a \otimes b$

Example 3 : mildly abstract description of BGP's ASPATHs

Let $\text{apaths}(X) = (\mathcal{E}(\Sigma^*) \cup \{\infty\},\ \Sigma \times \Sigma,\ \triangleright)$ where

$$
\begin{array}{rcl}
\mathcal{E}(\Sigma^*) & = & \text{finite, elementary sequences over } \Sigma \text{ (no repeats)} \\
(m,\ n)\ \triangleright\ \infty & = & \infty \\
(m,\ n)\ \triangleright\ l & = & \left\{ \begin{array}{ll} n \cdot l & (\text{if } m \notin n \cdot l) \\ \infty & (\text{otherwise}) \end{array} \right.
\end{array}
$$

HW2 — Due 27 November

1. Construct an interesting example using the *semi-direct product*.
2. Construct an interesting example using the *scoped product*.
3. Let **A** be an adjacency matrix for a directed graph G weighted over semiring S. Let **B** be a matrix over the Semiring of Elementary Paths $\mathrm{sep}(G)$ such that

$$\begin{array}{lcl} \mathbf{A}(i,j) = \overline{0} & \Longleftrightarrow & \mathbf{B}(i,j) = \{\} \\ \mathbf{A}(i,j) \neq \overline{0} & \Longleftrightarrow & \mathbf{B}(i,j) = \{(i,j)\} \end{array}$$

Is it always the case that

$$\mathbf{A}^*(i,j) = \bigoplus_{p \in \mathbf{B}^*(i,j)} w(p)$$

holds?

Here "an interesting example" means a specific algebraic structure, a graph weighted over that structure, either a global, left-, or right-local solutuon.