L11 : Algebraic Path Problems with Applications to Internet Routing
Lecture 10

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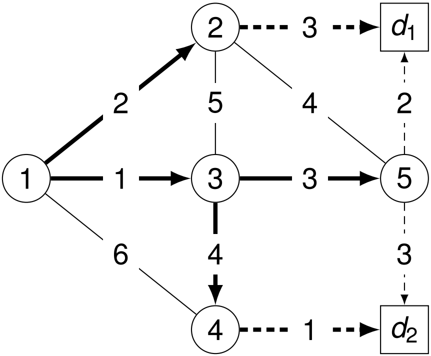
Example of routing = path finding + mapping



matrix	solves
$\mathbf{A}^*$	$\mathbf{R} = (\mathbf{A} \otimes \mathbf{R}) \oplus \mathbf{I}$
$\mathbf{A}^*\mathbf{M}$	$\mathbf{F} = (\mathbf{A} \otimes \mathbf{F}) \oplus \mathbf{M}$

$$\mathbf{M} \;=\; \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \left[\begin{array}{cc} \infty & \infty \\ 3 & \infty \\ \infty & \infty \\ \infty & 1 \\ 2 & 3 \end{array}\right] \end{array}$$

Mapping matrix

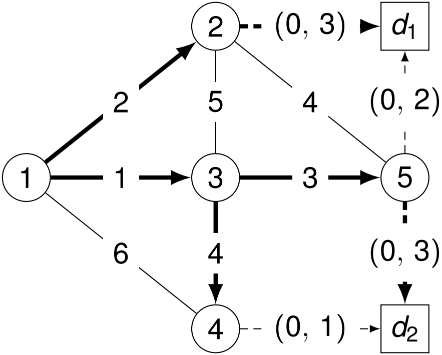
$$\mathbf{F} \;=\; \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \left[\begin{array}{cc} 5 & 6 \\ 3 & 7 \\ 5 & 5 \\ 9 & 1 \\ 2 & 3 \end{array}\right] \end{array}$$

Forwarding matrix

Routing Matrix vs. Path Matrix

- Inspired by the the Locator/ID split work
  - See Locator/ID Separation Protocol (LISP)
- Let's make a distinction between infrastructure nodes V and destinations D.
- Assume $V \cap D = \{\}$
- $\mathbf{M}$ is a $V \times D$ mapping matrix
  - $\mathbf{M}(v, d) \neq \infty$ means that destination (identifier) d is somehow attached to node (locator) v

More Interesting Example : Hot-Potato Idiom



$$
\mathbf{M} \;=\; \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}
\begin{array}{c} \begin{array}{cc} \quad d_1 \quad & \quad d_2 \quad \end{array} \\
\begin{bmatrix} \infty & \infty \\ (0, 3) & \infty \\ \infty & \infty \\ \infty & (0, 1) \\ (0, 2) & (0, 3) \end{bmatrix} \end{array}
$$

Mapping matrix

$$
\mathbf{F} \;=\; \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}
\begin{array}{c} \begin{array}{cc} \quad d_1 \quad & \quad d_2 \quad \end{array} \\
\begin{bmatrix} (2, 3) & (4, 3) \\ (0, 3) & (4, 3) \\ (3, 2) & (3, 3) \\ (7, 2) & (0, 1) \\ (0, 2) & (0, 3) \end{bmatrix} \end{array}
$$

Routing matrix

General Case

$G = (V, E)$, n is the size of V.

A $n \times n$ (left) path matrix $\mathbf{L}$ solves an equation of the form

$$\mathbf{L} = (\mathbf{A} \otimes \mathbf{L}) \oplus \mathbf{I},$$

over semiring S.

D is a set of destinations, with size d.

A $n \times d$ routing matrix is defined as

$$\mathbf{F} = \mathbf{L} \rhd \mathbf{M},$$

over some structure $(N, \ \Box, \ \rhd)$, where $\rhd \in S \to (N \to N)$.

routing $=$ path finding $+$ mapping

Does this make sense?

$$\mathbf{F}(i,\ d) = (\mathbf{L} \rhd \mathbf{M})(i,\ d) = \Box_{q \in V} \mathbf{L}(i,\ q) \rhd \mathbf{M}(q,\ d).$$

- Once again we are leaving paths implicit in the construction.
- Routing paths are best paths to egress nodes, selected with respect $\Box$-minimality.
- $\Box$-minimality can be very different from selection involved in path finding.

When we are lucky ...

matrix	solves
$\mathbf{A}^*$	$\mathbf{L} = (\mathbf{A} \otimes \mathbf{L}) \oplus \mathbf{I}$
$\mathbf{A}^* \rhd \mathbf{M}$	$\mathbf{F} = (\mathbf{A} \rhd \mathbf{F}) \Box \mathbf{M}$

When does this happen?

When $(N, \ \Box, \ \rhd)$ is a (left) semi-module over the semiring S.

(left) Semi-modules

- $(S, \oplus, \otimes, \overline{0}, \overline{1})$ is a semiring.

A (left) semi-module over S

Is a structure $(N, \Box, \rhd, \overline{0}_N)$, where

- $(N, \Box, \overline{0}_N)$ is a commutative monoid
- $\rhd$ is a function $\rhd \in (S \times N) \to N$
- $(a \otimes b) \rhd m = a \rhd (b \rhd m)$
- $\overline{0} \rhd m = \overline{0}_N$
- $s \rhd \overline{0}_N = \overline{0}_N$
- $\overline{1} \rhd m = m$

and distributivity holds,

$$\begin{array}{lclcl} \mathrm{LD} & : & s \rhd (m \Box n) & = & (s \rhd m) \Box (s \rhd n) \\ \mathrm{RD} & : & (s \oplus t) \rhd m & = & (s \rhd m) \Box (t \rhd m) \end{array}$$

Example : Hot-Potato

S idempotent and selective

$$\begin{array}{rcl} S & = & (S,\ \oplus_S,\ \otimes_S) \\ T & = & (T,\ \oplus_T,\ \otimes_T) \\ \rhd_{\mathrm{fst}} & \in & S \to (S \times T) \to (S \times T) \\ s_1 \rhd_{\mathrm{fst}} (s_2,\ t) & = & (s_1 \otimes_S s_2,\ t) \end{array}$$

$$\mathrm{Hot}(S,\ T) = (S \times T,\ \vec{\oplus},\ \rhd_{\mathrm{fst}}),$$

where $\vec{\oplus}$ is the (left) lexicographic product of $\oplus_S$ and $\oplus_T$.

Define $\rhd_{\mathrm{hp}}$ on matrices

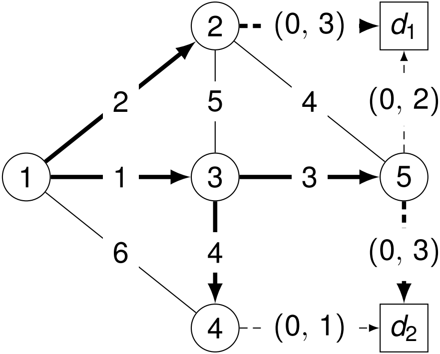
$$(\mathbf{L} \rhd_{\mathrm{hp}} \mathbf{M})(i,\ d) = \vec{\bigoplus}_{q \in V} \mathbf{L}(i,\ q) \rhd_{\mathrm{fst}} \mathbf{M}(q,\ d)$$

Sanity Check : does this implement hot-potato?

Define M to be simple if either $\mathbf{M}(v,\ d) = (1_S,\ t)$ or $\mathbf{M}(v,\ d) = (\infty_S,\ \infty_T)$.

$$
\begin{aligned}
& (\mathbf{L} \rhd_{\mathrm{hp}} \mathbf{M})(i,\ d) \\
= & \sum_{q \in V}^{\vec{\oplus}} \mathbf{L}(i,\ q) \rhd_{\mathrm{fst}} \mathbf{M}(q,\ d) \\
= & \sum_{\substack{q \in V \\ \mathbf{M}(q,\ d) = (s,\ t)}}^{\vec{\oplus}} (\mathbf{L}(i,\ q) \otimes_S s,\ t) \\
= & \sum_{\substack{q \in V \\ \mathbf{M}(q,\ d) = (1_S,\ t)}}^{\vec{\oplus}} (\mathbf{L}(i,\ q),\ t) \qquad \text{(if } M \text{ is simple)}
\end{aligned}
$$

Example of *hot-potato* routing



$$\mathbf{M} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \begin{bmatrix} \infty & \infty \\ (0,3) & \infty \\ \infty & \infty \\ \infty & (0,1) \\ (0,2) & (0,3) \end{bmatrix} \end{array}$$

Mapping matrix

$$\mathbf{F} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \begin{bmatrix} (2,3) & (4,3) \\ (0,3) & (4,3) \\ (3,2) & (3,3) \\ (7,2) & (0,1) \\ (0,2) & (0,3) \end{bmatrix} \end{array}$$

Routing matrix

matrix	solves
$\mathbf{A}^*$	$\mathbf{L} = (\mathbf{A} \otimes \mathbf{L}) \oplus \mathbf{I}$
$\mathbf{A}^* \rhd_{\mathrm{hp}} \mathbf{M}$	$\mathbf{F} = (\mathbf{A} \rhd_{\mathrm{hp}} \mathbf{F}) \vec{\oplus} \mathbf{M}$

Example : Cold-Potato

T idempotent and selective

$$\begin{array}{rcl} S & = & (S, \oplus_S, \otimes_S) \\ T & = & (T, \oplus_T, \otimes_T) \\ \rhd_{\mathrm{fst}} & \in & S \to (S \times T) \to (S \times T) \\ s_1 \rhd_{\mathrm{fst}} (s_2, t) & = & (s_1 \otimes_S s_2, t) \end{array}$$

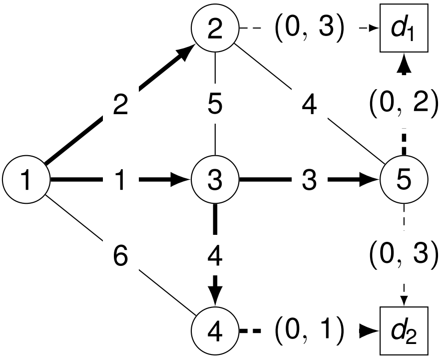
$$\mathrm{Cold}(S, T) = (S \times T, \overleftarrow{\oplus}, \rhd_{\mathrm{fst}}),$$

where $\vec{\oplus}$ is the (left) lexicographic product of $\oplus_S$ and $\oplus_T$.

Define $\rhd_{\mathrm{cp}}$ on matrices

$$(\mathbf{L} \rhd_{\mathrm{cp}} \mathbf{M})(i, d) = \overleftarrow{\bigoplus}_{q \in V} \mathbf{L}(i, q) \rhd_{\mathrm{fst}} \mathbf{M}(q, d)$$

Example of *cold-potato* routing



$$\mathbf{M} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cc} \quad d_1 \quad & \quad d_2 \quad \end{array} \\ \begin{bmatrix} \infty & \infty \\ (0, 3) & \infty \\ \infty & \infty \\ \infty & (0, 1) \\ (0, 2) & (0, 3) \end{bmatrix} \end{array}$$

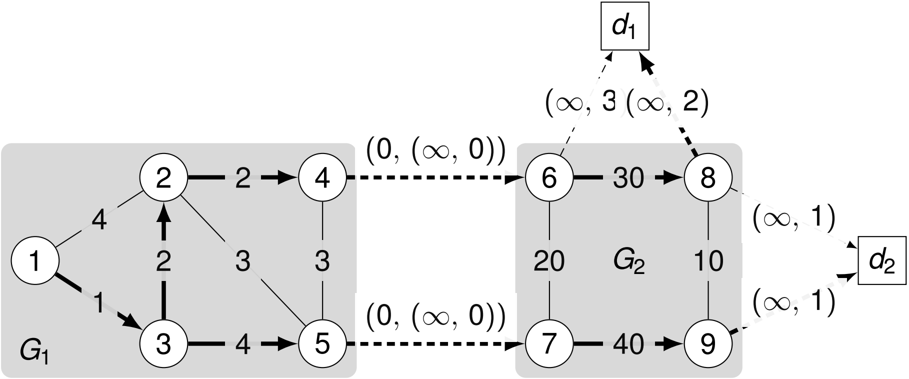
Mapping matrix

$$\mathbf{F} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cc} \quad d_1 \quad & \quad d_2 \quad \end{array} \\ \begin{bmatrix} (4, 2) & (5, 1) \\ (4, 2) & (9, 1) \\ (3, 2) & (4, 1) \\ (7, 2) & (0, 1) \\ (0, 2) & (7, 1) \end{bmatrix} \end{array}$$

Routing matrix

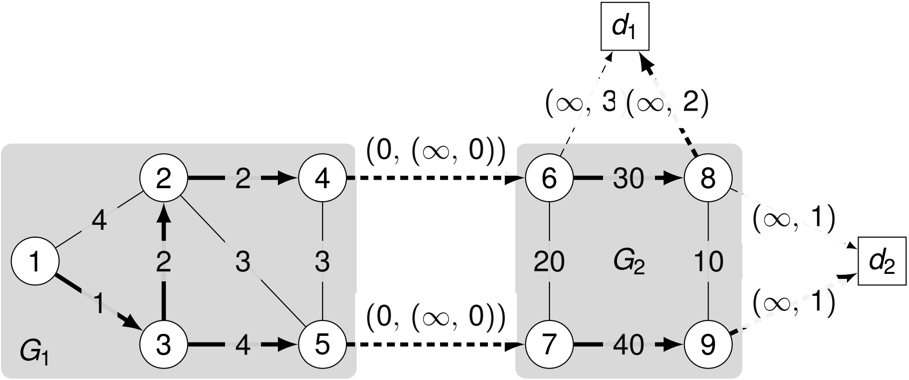
matrix	solves
$\mathbf{A}^*$	$\mathbf{L} = (\mathbf{A} \otimes \mathbf{L}) \oplus \mathbf{I}$
$\mathbf{A}^* \rhd_{\mathrm{cp}} \mathbf{M}$	$\mathbf{F} = \mathbf{A} \rhd_{\mathrm{cp}} \mathbf{F} \overleftarrow{\oplus} \mathbf{M}$

A simple example of route redistribution



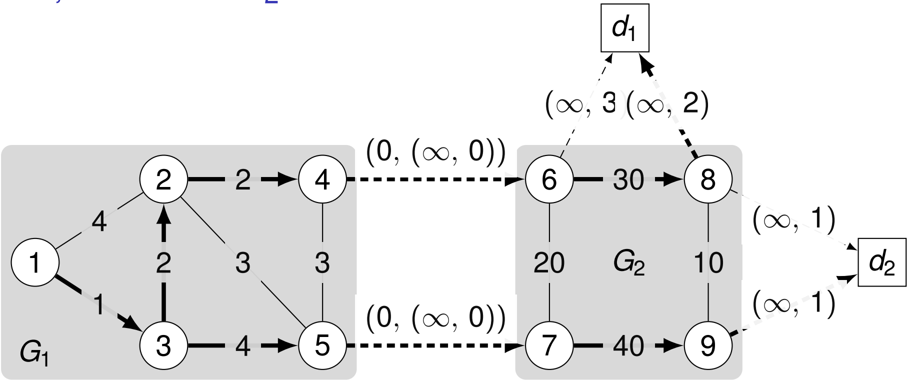
We will will use the routing of G_2 to construct a mapping for G_1 ...

A simple example of route redistribution



- G_2 is path finding with the bandwidth semiring bw
- G_2 is routing with $\mathrm{Cold}(\mathrm{bw},\ \mathrm{sp})$
- G_1 is path finding with the bandwidth semiring sp
- G_1 is routing with $\mathrm{Hot}(\mathrm{sp},\ \mathrm{Cold}(\mathrm{bw},\ \mathrm{sp}))$

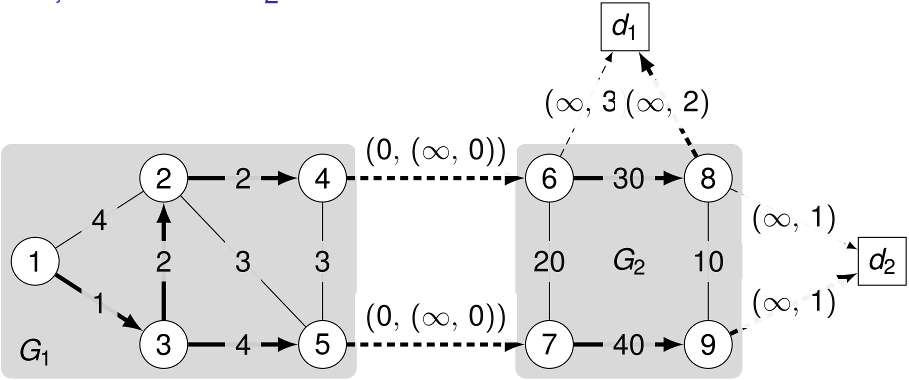
First, construct $\mathbf{F}_2$



$$\mathbf{L}_2 = \begin{array}{c} \\ 6 \\ 7 \\ 8 \\ 9 \end{array} \begin{array}{c} \begin{array}{cccc} 6 & 7 & 8 & 9 \end{array} \\ \begin{bmatrix} \infty & 20 & 30 & 20 \\ 20 & \infty & 20 & 40 \\ 30 & 20 & \infty & 20 \\ 20 & 40 & 20 & \infty \end{bmatrix} \end{array}$$

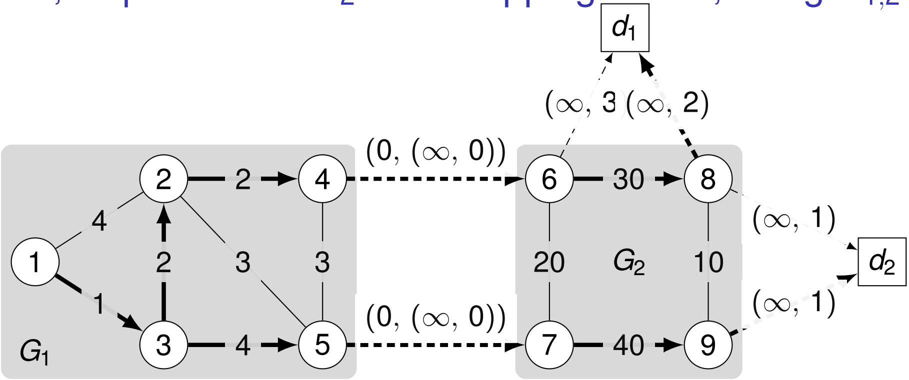
$$\mathbf{M}_2 = \begin{array}{c} \\ 6 \\ 7 \\ 8 \\ 9 \end{array} \begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \begin{bmatrix} (\infty, 3) & \infty \\ \infty & \infty \\ (\infty, 2) & (\infty, 1) \\ \infty & (\infty, 1) \end{bmatrix} \end{array}$$

First, construct $\mathbf{F}_2$



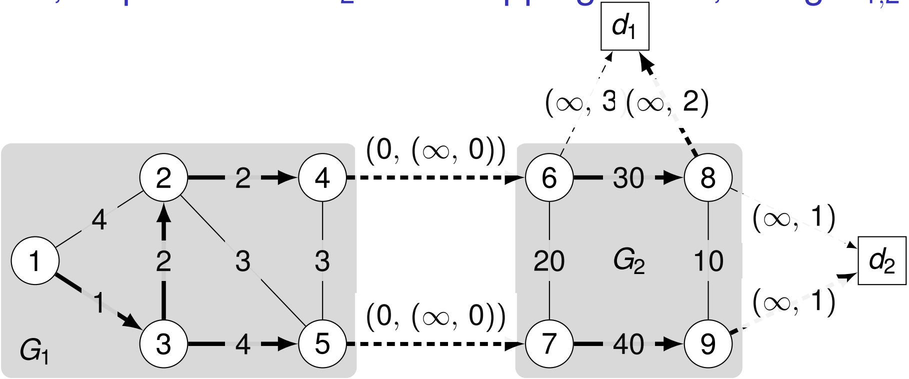
$$\mathbf{F}_2 = \mathbf{L}_2 \rhd_{\mathrm{cp}} \mathbf{M}_2 = \begin{array}{c} \\ 6 \\ 7 \\ 8 \\ 9 \end{array} \begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \begin{bmatrix} (30, 2) & (30, 1) \\ (20, 2) & (40, 1) \\ (\infty, 2) & (\infty, 1) \\ (20, 2) & (\infty, 1) \end{bmatrix} \end{array}$$

Now, ship it over to G_2 as a mapping matrix, using $\mathbf{B}_{1,2}$



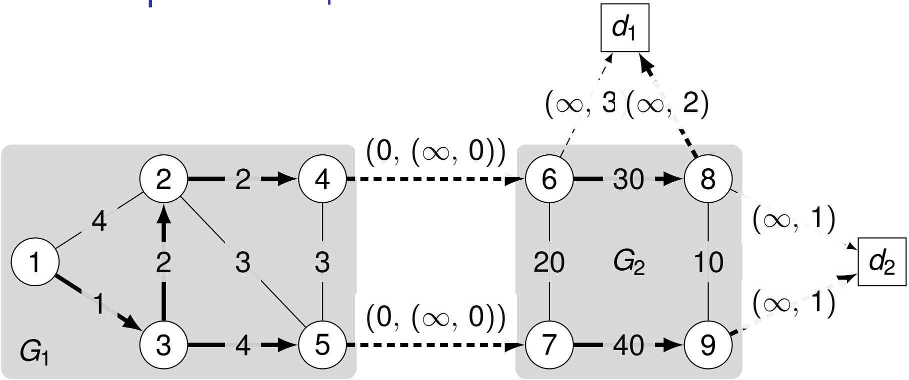
$$\mathbf{B}_{1,2} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccc} 6 & 7 & 8 & 9 \end{array} \\ \begin{bmatrix} \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty \\ (0, (\infty, 0)) & \infty & \infty & \infty \\ \infty & (0, (\infty, 0)) & \infty & \infty \end{bmatrix} \end{array}$$

Now, ship it over to G_2 as a mapping matrix, using $\mathbf{B}_{1,2}$



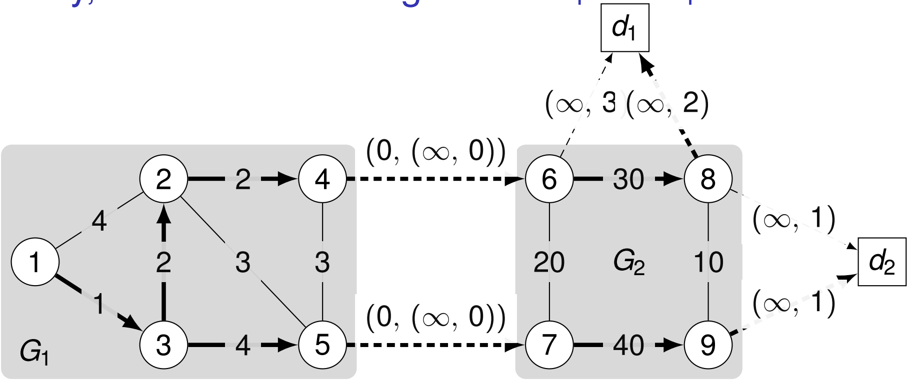
$$\mathbf{M}_1 = \mathbf{B}_{1,2} \lhd_{\text{hp}} \mathbf{F}_2 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cc} \quad d_1 \quad & \quad d_2 \quad \end{array} \\ \begin{bmatrix} \infty & \infty \\ \infty & \infty \\ \infty & \infty \\ (0, (30, 2)) & (0, (30, 1)) \\ (0, (20, 2)) & (0, (40, 1)) \end{bmatrix} \end{array}$$

Find best paths in G_1



$$\mathbf{L}_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 3 & 1 & 5 & 5 \\ 3 & 0 & 2 & 2 & 3 \\ 1 & 2 & 0 & 4 & 4 \\ 5 & 2 & 4 & 0 & 3 \\ 5 & 3 & 4 & 3 & 0 \end{bmatrix} \end{array}$$

Finally, construct a routing matrix $\mathbf{F}_1$ for G_1



$$\mathbf{F}_1 = \mathbf{L}_1 \rhd_{\text{hp}} \mathbf{M}_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cc} \quad d_1 \quad & \quad d_2 \quad \end{array} \\ \left[\begin{array}{cc} (5, (30, 2)) & (5, (40, 1)) \\ (2, (30, 2)) & (2, (30, 1)) \\ (4, (30, 2)) & (4, (40, 1)) \\ (0, (30, 2)) & (0, (30, 1)) \\ (0, (20, 2)) & (0, (40, 1)) \end{array}\right] \end{array}$$