L11 : Algebraic Path Problems with Applications to Internet Routing
Lectures 7 and 8

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Shortest paths with discounting

See page 51 of Baras and Theodorakopoulos.

Define $((\mathbb{R} \times \mathbb{N}) \cup \{\infty\},\ \oplus,\ \otimes)$ with

$$\oplus = \text{lexicographic on } \min_{\mathbb{R}} \text{ then } \min_{\mathbb{N}}$$

$$(r_1,\ n_1) \otimes (r_2,\ n_2) = (r_1 + r_2\delta^{n_1},\ n_1 + n_2)$$

where δ is a fixed constant between 0 and 1.

Suppose $w(i,j) = (\alpha(i,j),\ 1)$ and

$$p = (v_0, v_1)(v_1, v_2)\cdots(v_{k-1}, v_k).$$

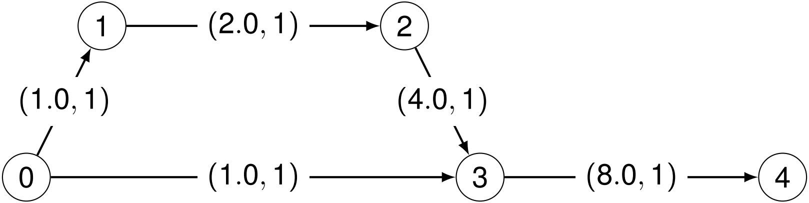
Then

$$w(p) = (\alpha,\ k)$$

where

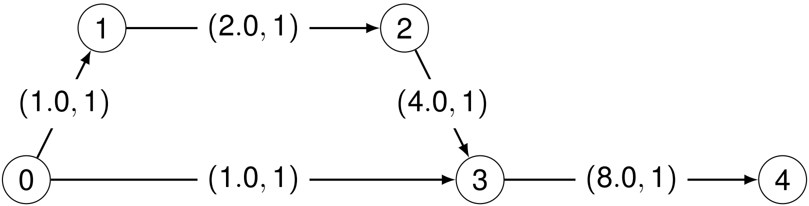
$$\alpha = \sum_{i=1}^{k} \delta^{i-1}\alpha(v_{i-1},\ v_i)$$

Example with $\delta = 0.5$ (part 1)



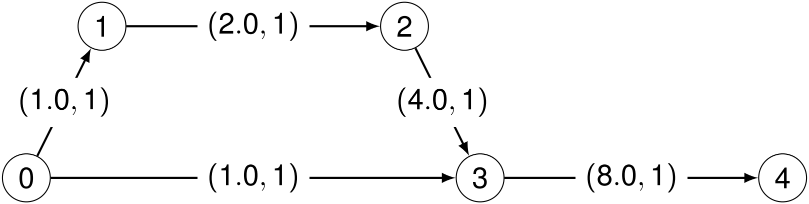
$$
\mathbf{A} \;=\; \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array}
\begin{array}{c}
\begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\
\left[\begin{array}{ccccc}
\infty & (1.0, 1) & \infty & (1.0, 1) & \infty \\
\infty & \infty & (2.0, 1) & \infty & \infty \\
\infty & \infty & \infty & (4.0, 1) & \infty \\
\infty & \infty & \infty & \infty & (8.0, 1) \\
\infty & \infty & \infty & \infty & \infty
\end{array}\right]
\end{array}
$$

Example with $\delta = 0.5$ (part 2)



$$
\begin{aligned}
w([(0,1)(1,2)]) &= (2.0,\ 2,\ ) \\
w([(0,1)(1,2)(2,3)]) &= (3.0,\ 3) \\
w([(0,1)(1,2)(2,3)(3,4)]) &= (4.0,\ 4) \\
w([(0,3)(3,4)]) &= (5.0,\ 2) \\
w([(1,2)(2,3)]) &= (4.0,\ 2) \\
w([(1,2)(2,3)(3,4)]) &= (6.0,\ 3) \\
w([(2,3)(3,4)]) &= (8.0,\ 2) \\
w([(3,4)]) &= (8.0,\ 1)
\end{aligned}
$$

Example with $\delta = 0.5$ (part 3)

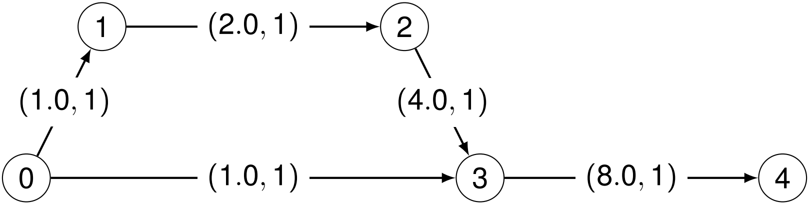


Observation: (right) distributivity does not hold!

$$
\begin{aligned}
& (w([(0,3)]) \oplus w([(0,1)(1,2)(2,3)])) \otimes w([(3,4)]) \\
= \; & ((1.0,\ 1) \oplus (3.0,\ 3)) \otimes (8.0,\ 1) \\
= \; & (1.0,\ 1) \otimes (8.0,\ 1) \\
= \; & (5.0,\ 2)
\end{aligned}
$$

$$
\begin{aligned}
& (w([(0,3)]) \otimes w([(3,4)])) \oplus (w([(0,1)(1,2)(2,3)]) \otimes w([(3,4)])) \\
= \; & ((1.0,\ 1) \otimes (8.0,\ 1)) \oplus ((3.0,\ 3) \otimes (8.0,\ 1)) \\
= \; & (5.0,\ 2) \oplus (4.0,\ 4) \\
= \; & (4.0,\ 4)
\end{aligned}
$$

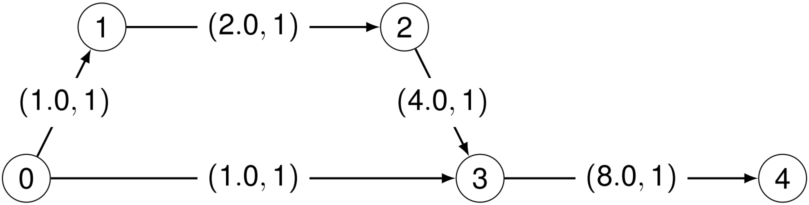
Example with $\delta = 0.5$ (part 4)



$$\mathbf{L} = \mathbf{AL} \oplus \mathbf{I}$$

	0	1	2	3	4
0	$(0.0, 0)$	$(1.0, 1)$	$(2.0, 2)$	$(1.0, 1)$	$(4.0, 4)$
1	∞	$(0.0, 0)$	$(2.0, 1)$	$(4.0, 2)$	$(6.0, 3)$
2	∞	∞	$(0.0, 0)$	$(4.0, 1)$	$(8.0, 2)$
3	∞	∞	∞	$(0.0, 0)$	$(8.0, 1)$
4	∞	∞	∞	∞	$(0.0, 0)$

Example with $\delta = 0.5$ (part 5)



$$\mathbf{R} = \mathbf{RA} \oplus \mathbf{I}$$

	0	1	2	3	4
0	(0.0, 0)	(1.0, 1)	(2.0, 2)	(1.0, 1)	(5.0, 2)
1	∞	(0.0, 0)	(2.0, 1)	(4.0, 2)	(6.0, 3)
2	∞	∞	(0.0, 0)	(4.0, 1)	(8.0, 2)
3	∞	∞	∞	(0.0, 0)	(8.0, 1)
4	∞	∞	∞	∞	(0.0, 0)

What's going on here?

We need to understand the consequences of a loss of distributivity.

But first, we will try to understand how *shortest paths with discounting* can be seen as a special case of a more general constructions — the semi-direct product.

Next lecture we will start to look more closely at the consequences of a loss of distributivity ...

Semi-direct product

Given semigroups $(S, \star)$ and $(T, \bullet)$, and a function

$$\rhd \in T \rightarrow (S \rightarrow S)$$

Define combinator $(S, \star) \rtimes (T, \bullet) = (S \times T, \rtimes)$

$$(s_1, t_1) \rtimes (s_2, t_2) = (s_1 \star (t_1 \rhd s_2), t_1 \bullet t_2)$$

Write $t \rhd s$ instead of $\rhd(t)(s)$.

What about associativity?

Requires $\rhd$ to be a homomorphism that maps to endomphisms.

homomorphism $\forall s\ t_1\ t_2,\ (t_1 \bullet t_2) \rhd s = t_1 \rhd (t_2 \rhd s)$

$(\forall t_1\ t_2,\ \rhd(t_1 \bullet t_2) = (\rhd t_1) \circ (\rhd t_2))$

endomorphism $\forall s_1\ s_2\ t,\ t \rhd (s_1 \star s_2) = (t \rhd s_1) \star (t \rhd s_2)$

homomorphism	$\rhd$ is a homomorhism from $(T,\ \bullet)$ to $(S \rightarrow S,\ \circ)$.
endomorphism	for every t, $f(s) = t \rhd s$ is homomorhism from $(S,\ \star)$ to $(S,\ \star)$.

Proof of associativity

$$
\begin{aligned}
\text{lhs} &= ((s_1,\ t_1)\ \rtimes\ (s_2,\ t_2))\ \rtimes\ (s_3,\ t_3) \\
&= (s_1 \star (t_1 \rhd s_2),\ t_1 \bullet t_2)\ \rtimes\ (s_3,\ t_3) \\
&= ((s_1 \star (t_1 \rhd s_2)) \star ((t_1 \bullet t_2) \rhd s_3),\ (t_1 \bullet t_2) \bullet t_3) \\
&= ((s_1 \star (t_1 \rhd s_2)) \star (t_1 \rhd (t_2 \rhd s_3)),\ (t_1 \bullet t_2) \bullet t_3) \quad \text{(by \textbf{homo})}
\end{aligned}
$$

$$
\begin{aligned}
\text{rhs} &= (s_1,\ t_1)\ \rtimes\ ((s_2,\ t_2)\ \rtimes\ (s_3,\ t_3)) \\
&= (s_1,\ t_1)\ \rtimes\ (s_2 \star (t_2 \rhd s_3),\ t_2 \bullet t_3) \\
&= (s_1 \star (t_1 \rhd (s_2 \star (t_2 \rhd s_3))),\ t_1 \bullet (t_2 \bullet t_3)) \\
&= (s_1 \star ((t_1 \rhd s_2) \star (t_1 \rhd (t_2 \rhd s_3))),\ t_1 \bullet (t_2 \bullet t_3)) \quad \text{(by \textbf{endo})}
\end{aligned}
$$

Now use associativity of $\star$ and $\bullet$.

Example!

$$
\begin{aligned}
\delta &\in (0, 1) \\
(S, \star) &= (\mathbb{R}, +) \\
(T, \bullet) &= (\mathbb{N}, +) \\
n \rhd c &= c \times \delta^n
\end{aligned}
$$

$S \rtimes T = (\mathbb{R} \times \mathbb{N}, \rtimes)$

$$(c_1, n_1) \rtimes (c_2, n_2) = (c_1 + c_2\delta^{n_1}, n_1 + n_2)$$

This is the multiplicative component of shortest paths with discounting.

Associative? Yes!

homomorphism $\forall c\ n_1\ n_2,\ c\delta^{n_1+n_2} = c\delta^{n_1}\delta^{n_2}$

endomorphism $\forall c_1\ c_2\ n,\ (c_1 + c_2)\delta^n = c_1\delta^n + c_2\delta^n$

When does $S \rtimes T$ have an identity?

Requires

$$
\begin{array}{rl}
(S,\ \star,\ 1_S) & \text{is a monoid} \\
(T,\ \bullet,\ 1_T) & \text{is a monoid} \\
\textbf{homo}\text{morphism} & \forall s, 1_T \rhd s = s \\
\textbf{endo}\text{morphism} & \forall t, t \rhd 1_S = 1_s
\end{array}
$$

Then $1 = (1_S,\ 1_T)$ is the identity.

$$
\begin{aligned}
(s,\ t) \rtimes (1_S,\ 1_T) &= (s \star (t \rhd 1_S),\ t \bullet 1_T) \\
&= (s \star 1_S,\ t) \\
&= (s,\ t) \\
(1_S,\ 1_T) \rtimes (s,\ t) &= (1_S \star (1_T \rhd s),\ 1_T \bullet t) \\
&= (1_S \star s,\ t) \\
&= (s,\ t)
\end{aligned}
$$

When does $S \rtimes T$ have an annihilator?

Requires

0_S	is the annihilator for S
0_T	is the annihilator for T
endomorphism	$\forall t, t \rhd 0_S = 0_s$

Then $0 = (0_S,\ 0_T)$ is the annihilator.

$$
\begin{aligned}
(s,\ t) \rtimes (0_S,\ 0_T) &= (s \star (t \rhd 0_S),\ t \bullet 0_T) \\
&= (s \star 0_S,\ 0_T) \\
&= (0_S,\ 0_T) \\
(0_S,\ 0_T) \rtimes (s,\ t) &= (0_S \star (0_T \rhd s),\ 0_T \bullet t) \\
&= (0_S,\ 0_T)
\end{aligned}
$$

Extend to combinators on bisemigroups?

Assume $(S, \oplus)$ is *selective*. Given bisemigroups $(S,\ \oplus,\ \star)$ and $(T,\ \boxplus,\ \bullet)$, and function

$$\rhd \in T \rightarrow (S \rightarrow S)$$

Try lexicographic addition ...

$$(S,\ \oplus,\ \star)\ \rtimes_{\rhd}\ (T,\ \boxplus,\ \bullet) = (S \times T,\ +,\ \rtimes)$$

$$(s_1,\ t_1) + (s_2,\ t_2) \ = \ \begin{cases} (s_1,\ t_1 \boxplus t_2) & (\text{if } s_1 = s_1) \\ (s_1,\ t_1) & (\text{if } s_1 = s_1 \oplus s_2 \neq s_2) \\ (s_2,\ t_2) & (\text{if } s_1 \neq s_1 \oplus s_2 = s_2) \end{cases}$$

$$(s_1,\ t_1)\ \rtimes\ (s_2,\ t_2) \ = \ (s_1 \star (t_1 \rhd s_2),\ t_1 \bullet t_2)$$

Left Distributive? Sometimes...

Here are some sufficient conditions

$$\begin{array}{rl} \star\text{-cancel} & \forall s\ s_1\ s_2 \in S,\ s \star s_1 = s \star s_2 \implies s_1 = s_2 \\ \rhd\text{-cancel} & \forall t \in T,\ s_1\ s_2 \in S,\ t \rhd s_1 = t \rhd s_2 \implies s_1 = s_2 \\ \rhd\text{-distribute} & \forall t \in T,\ s_1\ s_2 \in S,\ t \rhd (s_1 \oplus s_2) = (t \rhd s_1) \oplus (t \rhd s_2) \end{array}$$

Note that these hold in sp with discounting

$$\begin{array}{rl} \star\text{-cancel} & \forall s\ s_1\ s_2 \in \mathbb{R},\ s + s_1 = s + s_2 \implies s_1 = s_2 \\ \rhd\text{-cancel} & \forall t \in \mathbb{N},\ s_1\ s_2 \in \mathbb{R},\ s_1\delta^t = s_2\delta^t \implies s_1 = s_2 \\ \rhd\text{-distribute} & \forall t \in \mathbb{N},\ s_1\ s_2 \in \mathbb{R},\ (s_1 \min s_2)\delta^t = (s_1\delta^t) \min (s_2\delta^t) \end{array}$$

Left Distributive?

$$
\begin{aligned}
\text{lhs} &= ((s_1,\ t_1)\ \rtimes\ (s_2,\ t_2)) + ((s_1,\ t_1)\ \rtimes\ (s_3,\ t_3)) \\
&= (s_1 \star (t_1 \rhd s_2),\ t_1 \bullet t_2) + (s_1 \star (t_1 \rhd s_3),\ t_1 \bullet t_3) \\
&= ((s_1 \star (t_1 \rhd s_2)) \oplus (s_1 \star (t_1 \rhd s_3)),\ t) \\
&= (s_1 \star ((t_1 \rhd s_2) \oplus (t_1 \rhd s_3)),\ t) \\
&= (s_1 \star (t_1 \rhd (s_2 \oplus s_3)),\ t) \quad \text{(by } \rhd\text{-distribute)}
\end{aligned}
$$

$$
\begin{aligned}
\text{rhs} &= (s_1,\ t_1)\ \rtimes\ ((s_2,\ t_2) + (s_3,\ t_3)) \\
&= (s_1,\ t_1)\ \rtimes\ (s_2 \oplus s_3,\ t') \\
&= (s_1 \star (t_1 \rhd (s_2 \oplus s_3)),\ t_1 \bullet t')
\end{aligned}
$$

Left Distributive?

We want

$$t = t_1 \bullet t'$$

$$t_1 \bullet t' = \begin{cases} t_1 \bullet (t_2 \boxplus t_3) & (s_2 = s_3) \\ t_1 \bullet t_2 & (s_2 < s_3) \\ t_1 \bullet t_3 & (s_2 > s_3) \end{cases}$$

$$t = \begin{cases} (t_1 \bullet t_2) \boxplus (t_1 \bullet t_3) & (s_1 \star (t_1 \rhd s_2) = s_1 \star (t_1 \rhd s_3)) \\ t_1 \bullet t_2 & (s_1 \star (t_1 \rhd s_2) < s_1 \star (t_1 \rhd s_3)) \\ t_1 \bullet t_3 & (s_1 \star (t_1 \rhd s_2) > s_1 \star (t_1 \rhd s_3)) \end{cases}$$

Left Distributive?

Enough to show

$$\begin{array}{lcl} s < s' & \Longrightarrow & s'' \star s < s'' \star s' \\ s < s' & \Longrightarrow & t \rhd s < t \rhd s' \end{array}$$

We just show the second implication. Note that $\rhd$-cancel implies

$$\forall t \in T,\ s_1\ s_2 \in S,\ s_1 \neq s_2 \implies t \rhd s_1 \neq t \rhd s_2.$$

So

$$\begin{array}{lcl} s < s' & \Longrightarrow & s = s \oplus s' \neq s' \\ & \Longrightarrow & t \rhd s = t \rhd (s \oplus s') \neq t \rhd s' \\ & \Longrightarrow & t \rhd s = (t \rhd s) \oplus (t \rhd s') \neq t \rhd s' \\ & \Longrightarrow & t \rhd s < t \rhd s' \end{array}$$

Right Distributive? Almost never?

Assume both S and T are right distributive.

$$\begin{aligned}
\text{lhs} &= ((s_1,\ t_1) \rtimes (s_3,\ t_3)) + ((s_2,\ t_2) \rtimes (s_3,\ t_3)) \\
&= (s_1 \star (t_1 \rhd s_3),\ t_1 \bullet t_3) + (s_2 \star (t_2 \rhd s_3),\ t_2 \bullet t_3) \\
&= ((s_1 \star (t_1 \rhd s_3)) \oplus (s_2 \star (t_2 \rhd s_3)),\ t)
\end{aligned}$$

$$\begin{aligned}
\text{rhs} &= ((s_1,\ t_1) + (s_2,\ t_2)) \rtimes (s_3,\ t_3) \\
&= (s_1 \oplus s_2,\ t') \rtimes (s_3,\ t_3) \\
&= ((s_1 \oplus s_2) \star (t' \rhd s_3),\ t' \bullet t_3) \\
&= ((s_1 \star (t' \rhd s_3)) \oplus (s_2 \star (t' \rhd s_3)),\ t' \bullet t_3)
\end{aligned}$$

Right Distributive? Almost never?

We want

$$
\begin{array}{rcl}
t & = & t' \bullet t_3 \\
(s_1 \star (t_1 \rhd s_3)) \oplus (s_2 \star (t_2 \rhd s_3)) & = & (s_1 \star (t' \rhd s_3)) \oplus (s_2 \star (t' \rhd s_3))
\end{array}
$$

Where

$$
t' \bullet t_3 = \begin{cases} (t_1 \boxplus t_2) \bullet t_3 & (s_1 = s_2) \\ t_1 \bullet t_3 & (s_1 < s_2) \\ t_2 \bullet t_3 & (s_1 > s_2) \end{cases}
$$

and

$$
t = \begin{cases} (t_1 \bullet t_3) \boxplus (t_2 \bullet t_3) & (s_1 \star (t_1 \rhd s_3) = (s_2 \star (t_2 \rhd s_3))) \\ (t_1 \bullet t_3) & (s_1 \star (t_1 \rhd s_3) < (s_2 \star (t_2 \rhd s_3))) \\ (t_2 \bullet t_3) & (s_1 \star (t_1 \rhd s_3) > (s_2 \star (t_2 \rhd s_3))) \end{cases}
$$

OUCH! OUCH! OUCH! OUCH! OUCH! OUCH! OUCH!

Left-Local Optimality

Say that $\mathbf{L}$ is a left locally-optimal solution when

$$\mathbf{L} = (\mathbf{A} \otimes \mathbf{L}) \oplus \mathbf{I}.$$

That is, for $i \neq j$ we have

$$\mathbf{L}(i,\ j) = \bigoplus_{q \in V} \mathbf{A}(i,\ q) \otimes \mathbf{L}(q,\ j)$$

- $\mathbf{L}(i,\ j)$ is the best possible value given the values $\mathbf{L}(q,\ j)$, for all out-neighbors q of source i.
- Rows $\mathbf{L}(i,\ \_)$ represents **out-trees <u>from</u>** i (think Bellman-Ford).
- Columns $\mathbf{L}(\_,\ i)$ represents **in-trees <u>to</u>** i.
- Works well with hop-by-hop forwarding from i.

Right-Local Optimality

Say that $\mathbf{R}$ is a right locally-optimal solution when

$$\mathbf{R} = (\mathbf{R} \otimes \mathbf{A}) \oplus \mathbf{I}.$$

That is, for $i \neq j$ we have

$$\mathbf{R}(i,\ j) = \bigoplus_{q \in V} \mathbf{R}(i,\ q) \otimes \mathbf{A}(q,\ j)$$

- $\mathbf{R}(i,\ j)$ is the best possible value given the values $\mathbf{R}(q,\ j)$, for all in-neighbors q of destination j.
- Rows $\mathbf{L}(i,\ \_)$ represents **out-trees <u>from</u>** i (think Dijkstra).
- Columns $\mathbf{L}(\_,\ i)$ represents **in-trees <u>to</u>** i.
- Does not work well with hop-by-hop forwarding from i.

With and Without Distributivity

With

For semirings, the three optimality problems are essentially the same — locally optimal solutions are globally optimal solutions.

$$\mathbf{A}^* = \mathbf{L} = \mathbf{R}$$

Without

Suppose that we drop distributivity and $\mathbf{A}^*$, $\mathbf{L}$, $\mathbf{R}$ exist. It may be the case they they are all distinct.

Health warning : matrix multiplication over structures lacking distributivity is not associative!

With only left distributivity

Matrix powers, $\mathbf{A}^k$

$$\begin{aligned} \mathbf{A}^0 &= \mathbf{I} \\ \mathbf{A}^{k+1} &= \mathbf{A} \otimes \mathbf{A}^k \end{aligned}$$

Closure, $\mathbf{A}^*$

$$\begin{aligned} \mathbf{A}^{(k)} &= \mathbf{I} \oplus \mathbf{A}^1 \oplus \mathbf{A}^2 \oplus \cdots \oplus \mathbf{A}^k \\ \mathbf{A}^* &= \mathbf{I} \oplus \mathbf{A}^1 \oplus \mathbf{A}^2 \oplus \cdots \oplus \mathbf{A}^k \oplus \cdots \end{aligned}$$

Theorem

$$\mathbf{A}^k(i,\ j) = \bigoplus_{p \in P^k(i,\ j)} w(p)$$

With only right distributivity

Matrix powers, ${}^{k}\mathbf{A}$

$$
\begin{aligned}
{}^{0}\mathbf{A} &= \mathbf{I} \\
{}^{k+1}\mathbf{A} &= {}^{k}\mathbf{A} \otimes \mathbf{A}
\end{aligned}
$$

Closure, ${}^{*}\mathbf{A}$

$$
\begin{aligned}
{}^{(k)}\mathbf{A} &= \mathbf{I} \oplus {}^{1}\mathbf{A} \oplus {}^{2}\mathbf{A} \oplus \cdots \oplus {}^{k}\mathbf{A} \\
{}^{*}\mathbf{A} &= \mathbf{I} \oplus {}^{1}\mathbf{A} \oplus {}^{2}\mathbf{A} \oplus \cdots \oplus {}^{k}\mathbf{A} \oplus \cdots
\end{aligned}
$$

Theorem

$$
{}^{k}\mathbf{A}(i,\ j) = \bigoplus_{p \in P^{k}(i,\ j)} w(p)
$$