L11 : Algebraic Path Problems with Applications to Internet Routing
Lecture 6

Timothy G. Griffin

`timothy.griffin@cl.cam.ac.uk`
Computer Laboratory
University of Cambridge, UK

Michaelmas Term
2012

A note on $\mathbf{A}$ vs. $\mathbf{A} \oplus \mathbf{I}$

Lemma 6.0

If $\oplus$ is idempotent, then

$$(\mathbf{A} \oplus \mathbf{I})^k = \mathbf{A}^{(k)}.$$

Proof. Base case: When $k = 0$ both expressions are $\mathbf{I}$.
Assume $(\mathbf{A} \oplus \mathbf{I})^k = \mathbf{A}^{(k)}$. Then

$$\begin{array}{rcl}
(\mathbf{A} \oplus \mathbf{I})^{k+1} & = & (\mathbf{A} \oplus \mathbf{I})(\mathbf{A} \oplus \mathbf{I})^k \\
 & = & (\mathbf{A} \oplus \mathbf{I})\mathbf{A}^{(k)} \\
 & = & \mathbf{A}\mathbf{A}^{(k)} \oplus \mathbf{A}^{(k)} \\
 & = & \mathbf{A}(\mathbf{I} \oplus \mathbf{A} \oplus \cdots \oplus \mathbf{A}^k) \oplus \mathbf{A}^{(k)} \\
 & = & \mathbf{A} \oplus \mathbf{A}^2 \oplus \cdots \oplus \mathbf{A}^{k+1} \oplus \mathbf{A}^{(k)} \\
 & = & \mathbf{A}^{k+1} \oplus \mathbf{A}^{(k)} \\
 & = & \mathbf{A}^{(k+1)}
\end{array}$$

Solving (some) equations

Theorem 6.1

If $\mathbf{A}$ is q-stable, then $\mathbf{X} = \mathbf{A}^*$ solves the equations

$$\mathbf{X} = \mathbf{A}\mathbf{X} \oplus \mathbf{I}$$

and

$$\mathbf{X} = \mathbf{X}\mathbf{A} \oplus \mathbf{I}.$$

For example,

$$\begin{array}{rcl} \mathbf{A}^* & = & \mathbf{A}^{(q)} \\ & = & \mathbf{A}^{(q+1)} \\ & = & \mathbf{A}^{q+1} \oplus \mathbf{A}^{q} \oplus \ldots \oplus \mathbf{A}^2 \oplus \mathbf{A} \oplus \mathbf{I} \\ & = & \mathbf{A}(\mathbf{A}^{q} \oplus \mathbf{A}^{q-1} \oplus \ldots \oplus \mathbf{A} \oplus \mathbf{I}) \oplus \mathbf{I} \\ & = & \mathbf{A}\mathbf{A}^{(q)} \oplus \mathbf{I} \\ & = & \mathbf{A}\mathbf{A}^* \oplus \mathbf{I} \end{array}$$

Note that if we replace the assumption "**A** is *q*-stable" with "$\mathbf{A}^*$ exists," then we require that $\otimes$ distributes over <u>infinite</u> sums.

A more general result

Theorem 6.2

If $\mathbf{A}$ is q-stable, then $\mathbf{X} = \mathbf{A}^*\mathbf{B}$ solves the equations

$$\mathbf{X} = \mathbf{AX} \oplus \mathbf{B}$$

and $\mathbf{X} = \mathbf{BA}^*$ solves

$$\mathbf{X} = \mathbf{XA} \oplus \mathbf{I}.$$

For example,

$$\begin{aligned}
\mathbf{A}^*\mathbf{B} &= \mathbf{A}^{(q)}\mathbf{B} \\
&= \mathbf{A}^{(q+1)}\mathbf{B} \\
&= (\mathbf{A}^{q+1} \oplus \mathbf{A}^q \oplus \ldots \oplus \mathbf{A}^2 \oplus \mathbf{A} \oplus \mathbf{I})\mathbf{B} \\
&= (\mathbf{A}^{q+1} \oplus \mathbf{A}^q \oplus \ldots \oplus \mathbf{A}^2 \oplus \mathbf{A})\mathbf{B} \oplus \mathbf{B} \\
&= \mathbf{A}(\mathbf{A}^q \oplus \mathbf{A}^{q-1} \oplus \ldots \oplus \mathbf{A} \oplus \mathbf{I})\mathbf{B} \oplus \mathbf{B} \\
&= \mathbf{A}(\mathbf{A}^{(q)}\mathbf{B}) \oplus \mathbf{B} \\
&= \mathbf{A}(\mathbf{A}^*\mathbf{B}) \oplus \mathbf{B}
\end{aligned}$$

The "best" solution

Suppose $\mathbf{Y}$ is a matrix such that

$$\mathbf{Y} = \mathbf{A}\mathbf{Y} \oplus \mathbf{I}$$

$$\begin{array}{rcl}
\mathbf{Y} & = & \mathbf{A}\mathbf{Y} \oplus \mathbf{I} \\
 & = & \mathbf{A}^1\mathbf{Y} \oplus \mathbf{A}^{(0)} \\
 & = & \mathbf{A}((\mathbf{A}\mathbf{Y} \oplus \mathbf{I})) \oplus \mathbf{I} \\
 & = & \mathbf{A}^2\mathbf{Y} \oplus \mathbf{A} \oplus \mathbf{I} \\
 & = & \mathbf{A}^2\mathbf{Y} \oplus \mathbf{A}^{(1)} \\
 \vdots & & \vdots \quad \vdots \\
 & = & \mathbf{A}^{k+1}\mathbf{Y} \oplus \mathbf{A}^{(k)}
\end{array}$$

If $\mathbf{A}$ is q-stable and $q < k$, then

$$\mathbf{Y} = \mathbf{A}^k\mathbf{Y} \oplus \mathbf{A}^*$$

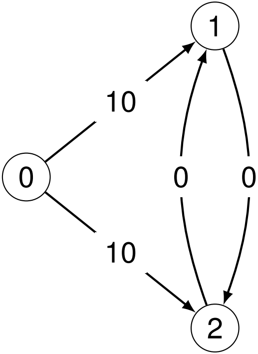
$$\mathbf{Y} \trianglelefteq^L_{\oplus} \mathbf{A}^*$$

and if $\oplus$ is idempotent, then

$$\mathbf{Y} \leq^L_{\oplus} \mathbf{A}^*$$

So $\mathbf{A}^*$ is the largest solution. What does this mean in terms of the sp semiring?

Example with zero weighted cycles using sp semiring



$$\mathbf{A} \quad = \quad \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & \infty & 10 & 10 \\ 1 & \infty & \infty & 0 \\ 2 & \infty & 0 & \infty \end{array}$$

$\mathbf{A}^*$ $(= \mathbf{A} \oplus \mathbf{I}$ in this case) solves

$$\mathbf{X} = \mathbf{X}\mathbf{A} \oplus \mathbf{I}.$$

But so does this (dishonest) matrix!

$$\mathbf{F} \quad = \quad \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0 & 9 & 9 \\ 1 & \infty & 0 & 0 \\ 2 & \infty & 0 & 0 \end{array}$$

For example :

$$\begin{array}{rl} & (\mathbf{FA})(0,1) \\ = & \min_{q\in\{0,1,2\}} \mathbf{F}(0,q) + \mathbf{A}(q,1) \\ = & \min(0+10, 9+\infty, 9+0) \\ = & 9 \\ = & \mathbf{F}(0,1) \end{array}$$

Recall our basic iterative algorithm

$$\begin{aligned} \mathbf{A}^{\langle 0 \rangle} &= \mathbf{I} \\ \mathbf{A}^{\langle k+1 \rangle} &= \mathbf{A}\mathbf{A}^{\langle k \rangle} \oplus \mathbf{I} \end{aligned}$$

A closer look ...

$$\begin{aligned} \mathbf{A}^{\langle k+1 \rangle}(i,j) &= \mathbf{I}(i,j) \oplus \bigoplus_{u} \mathbf{A}(i,u)\mathbf{A}^{\langle k \rangle}(u,j) \\ &= \mathbf{I}(i,j) \oplus \bigoplus_{(i,u) \in E} \mathbf{A}(i,u)\mathbf{A}^{\langle k \rangle}(u,j) \end{aligned}$$

This is the basis of distributed Bellman-Ford algorithms — a node *i* computes routes to a destination *j* by applying its link weights to the routes learned from its immediate neighbors. It then makes these routes available to its neighbors and the process continues...

What if we start iteration in an arbitrary state **M**?

In a distributed environment the topology (captured here by **A**) can change and the state of the computation can start in an arbitrary state (with respect to a new **A**).

$$\begin{aligned}\mathbf{A}_{\mathbf{M}}^{\langle 0\rangle} &= \mathbf{M}\\ \mathbf{A}_{\mathbf{M}}^{\langle k+1\rangle} &= \mathbf{A}\mathbf{A}_{\mathbf{M}}^{\langle k\rangle} \oplus \mathbf{I}\end{aligned}$$

Lemma 6.4

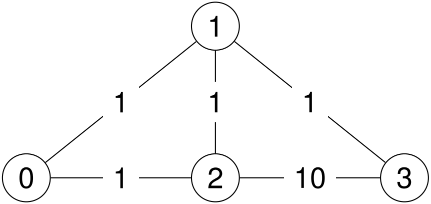
For $1 \leq k$,

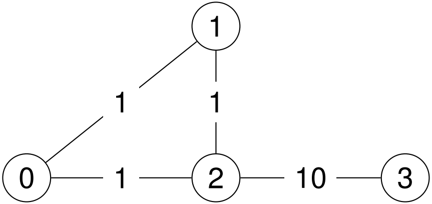
$$\mathbf{A}_{\mathbf{M}}^{\langle k\rangle} = \mathbf{A}^{k}\mathbf{M} \oplus \mathbf{A}^{(k-1)}$$

If **A** is q-stable and $q < k$, then

$$\mathbf{A}_{\mathbf{M}}^{\langle k\rangle} = \mathbf{A}^{k}\mathbf{M} \oplus \mathbf{A}^{*}$$

RIP-like example — counting to convergence (1)





Adjacency matrix $\mathbf{A}_1$

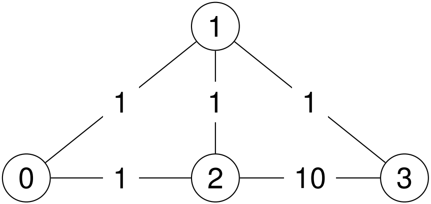
$$\begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & \infty & 1 & 1 & \infty \\ 1 & 1 & \infty & 1 & 1 \\ 2 & 1 & 1 & \infty & 10 \\ 3 & \infty & 1 & 10 & \infty \end{array}$$

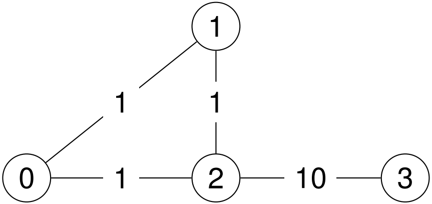
Adjacency matrix $\mathbf{A}_2$

$$\begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & \infty & 1 & 1 & \infty \\ 1 & 1 & \infty & 1 & \infty \\ 2 & 1 & 1 & \infty & 10 \\ 3 & \infty & \infty & 10 & \infty \end{array}$$

See RFC 1058.

RIP-like example — counting to convergence (2)





The solution $\mathbf{A}_1^*$

$$\begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 1 & 1 & 2 \\ 1 & 1 & 0 & 1 & 1 \\ 2 & 1 & 1 & 0 & 2 \\ 3 & 2 & 1 & 2 & 0 \end{array}$$

The solution $\mathbf{A}_2^*$

$$\begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 1 & 1 & 11 \\ 1 & 1 & 0 & 1 & 11 \\ 2 & 1 & 1 & 0 & 10 \\ 3 & 11 & 11 & 10 & 0 \end{array}$$

RIP-like example — counting to convergence (3)

The scenario: we arrived at $\mathbf{A}_1^*$, but then links $\{(1,3),\ (3,1)\}$ fail. So we start iterating using the new matrix $\mathbf{A}_2$.

Let $\mathbf{B}_K$ represent $\mathbf{A}_{2\,\mathbf{M}}^{\langle k \rangle}$, where $\mathbf{M} = \mathbf{A}_1^*$.

RIP-like example — counting to convergence (4)

$$\mathbf{B}_0 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix} \end{array}$$

$$\mathbf{B}_3 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 4 \\ 1 & 0 & 1 & 4 \\ 1 & 1 & 0 & 4 \\ 11 & 11 & 10 & 0 \end{bmatrix} \end{array}$$

$$\mathbf{B}_1 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 3 \\ 1 & 1 & 0 & 2 \\ 11 & 11 & 10 & 0 \end{bmatrix} \end{array}$$

$$\mathbf{B}_4 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 5 \\ 1 & 0 & 1 & 5 \\ 1 & 1 & 0 & 5 \\ 11 & 11 & 10 & 0 \end{bmatrix} \end{array}$$

$$\mathbf{B}_2 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 3 \\ 1 & 0 & 1 & 3 \\ 1 & 1 & 0 & 3 \\ 11 & 11 & 10 & 0 \end{bmatrix} \end{array}$$

$$\mathbf{B}_5 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 6 \\ 1 & 0 & 1 & 6 \\ 1 & 1 & 0 & 6 \\ 11 & 11 & 10 & 0 \end{bmatrix} \end{array}$$

RIP-like example — counting to convergence (5)

$$\mathbf{B}_6 = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 1 & 1 & 7 \\ 1 & 1 & 0 & 1 & 7 \\ 2 & 1 & 1 & 0 & 7 \\ 3 & 2 & 1 & 2 & 0 \end{array}$$

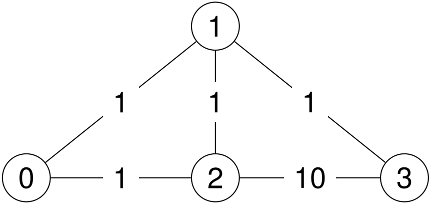
$$\mathbf{B}_9 = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 1 & 1 & 10 \\ 1 & 1 & 0 & 1 & 10 \\ 2 & 1 & 1 & 0 & 10 \\ 3 & 11 & 11 & 10 & 0 \end{array}$$

$$\mathbf{B}_7 = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 1 & 1 & 8 \\ 1 & 1 & 0 & 1 & 8 \\ 2 & 1 & 1 & 0 & 8 \\ 3 & 11 & 11 & 10 & 0 \end{array}$$

$$\mathbf{B}_{10} = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 1 & 1 & 11 \\ 1 & 1 & 0 & 1 & 11 \\ 2 & 1 & 1 & 0 & 10 \\ 3 & 11 & 11 & 10 & 0 \end{array}$$

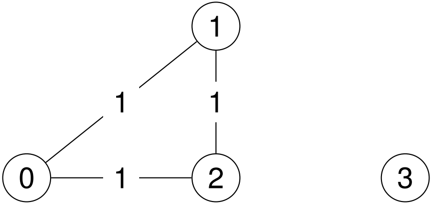
$$\mathbf{B}_8 = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 1 & 1 & 9 \\ 1 & 1 & 0 & 1 & 9 \\ 2 & 1 & 1 & 0 & 9 \\ 3 & 11 & 11 & 10 & 0 \end{array}$$

RIP-like example — counting to infinity (1)



The solution $\mathbf{A}_1^*$

$$\begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 1 & 1 & 2 \\ 1 & 1 & 0 & 1 & 1 \\ 2 & 1 & 1 & 0 & 2 \\ 3 & 2 & 1 & 2 & 0 \end{array}$$



The solution $\mathbf{A}_3^*$

$$\begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 1 & 1 & \infty \\ 1 & 1 & 0 & 1 & \infty \\ 2 & 1 & 1 & 0 & \infty \\ 3 & \infty & \infty & \infty & 0 \end{array}$$

Now let $\mathbf{B}_K$ represent $\mathbf{A}_{3\,\mathbf{M}}^{\langle k \rangle}$, where $\mathbf{M} = \mathbf{A}_1^*$.

RIP-like example — counting to infinity (2)

$$\mathbf{B}_0 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left[\begin{array}{cccc} 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 2 & 1 & 2 & 0 \end{array}\right] \end{array}$$

$$\mathbf{B}_1 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left[\begin{array}{cccc} 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 3 \\ 1 & 1 & 0 & 2 \\ \infty & \infty & \infty & 0 \end{array}\right] \end{array}$$

$$\mathbf{B}_2 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left[\begin{array}{cccc} 0 & 1 & 1 & 3 \\ 1 & 0 & 1 & 3 \\ 1 & 1 & 0 & 3 \\ \infty & \infty & \infty & 0 \end{array}\right] \end{array}$$

$$\vdots \quad \vdots \quad \vdots$$

$$\mathbf{B}_{376} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left[\begin{array}{cccc} 0 & 1 & 1 & 377 \\ 1 & 0 & 1 & 377 \\ 1 & 1 & 0 & 377 \\ \infty & \infty & \infty & 0 \end{array}\right] \end{array}$$

$$\vdots \quad \vdots \quad \vdots$$

$$\mathbf{B}_{998} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left[\begin{array}{cccc} 0 & 1 & 1 & 999 \\ 1 & 0 & 1 & 999 \\ 1 & 1 & 0 & 999 \\ \infty & \infty & \infty & 0 \end{array}\right] \end{array}$$

$$\vdots \quad \vdots \quad \vdots$$

RIP-like example — What's going on?

$$\mathbf{A}_{\mathbf{M}}^{\langle k \rangle} = \mathbf{A}^{k}\mathbf{M} \oplus \mathbf{A}^{*}$$

The $\mathbf{A}^{*}$ component may be arrived at very quickly but an entry of $\mathbf{A}^{k}\mathbf{M}$ may be better until a very large value of k is reached (counting to convergence) or it may always be better (counting to infinity).

RIP's solution? $\infty = 16$

BGP's solution? Combine with elementary paths!

BGP-like solution — path-vectoring (1)

Let's use our friend

$$\text{add\_zero}(\infty,\ \text{min\_plus}\ \vec{\times}\ \text{sep}(G))$$

Problem:

$$\vdots \quad \vdots \quad \vdots$$

$$\mathbf{B}_{998} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left[\begin{array}{cccc} (0, \{\epsilon\}) & (1, \{[(0,1)]\}) & (1, \{[(0,2)]\}) & (999, \{\}) \\ (1, \{[(1,0)]\}) & (0, \{\epsilon\}) & (1, \{[(1,2)]\}) & (999, \{\}) \\ (1, \{[(2,0)]\}) & (1, \{[(2,1)]\}) & (0, \{\epsilon\}) & (999, \{\}) \\ \infty & \infty & \infty & (0, \{\epsilon\}) \end{array}\right] \end{array}$$

$$\vdots \quad \vdots \quad \vdots$$

BGP-like solution — path-vectoring (2)

Solution: use another reduction!

$$r(s, W) = \begin{cases} \infty & \text{if } W = \{\} \\ (s, W) & \text{otherwise} \end{cases}$$

Now use this instead

$$\mathrm{red}_r(\mathrm{add\_zero}(\infty,\ \mathrm{min\_plus} \vec{\times} \mathrm{sep}(G)))$$

BGP-like solution — path-vectoring (3)

$\mathbf{B}_0$ and $\mathbf{B}_1$

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{cccc} 0 & 1 & 2 & 3 \\ \left[\begin{array}{c} (0, \{\epsilon\}) \\ (1, \{[(1,0)]\}) \\ (1, \{[(2,0)]\}) \\ (2, \{[(3,1),(1,0)]\}) \end{array}\right. & \begin{array}{c} (1, \{[(0,1)]\}) \\ (0, \{\epsilon\}) \\ (1, \{[(2,1)]\}) \\ (1, \{[(3,1)]\}) \end{array} & \begin{array}{c} (1, \{[(0,2)]\}) \\ (1, \{[(1,2)]\}) \\ (0, \{\epsilon\}) \\ (2, \{[(3,1),(1,2)]\}) \end{array} & \begin{array}{c} (2, \{[(0,1),(1,3 \\ (1, \{[(1,3)]\} \\ (2, \{[(2,1),(1,3 \\ (0, \{\epsilon\}) \end{array} \end{array}$$

$$\begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & (0, \{\epsilon\}) & (1, \{[(0,1)]\}) & (1, \{[(0,2)]\}) & (2, \{[(0,1),(1,3)]\}) \\ 1 & (1, \{[(1,0)]\}) & (0, \{\epsilon\}) & (1, \{[(1,2)]\}) & \infty \\ 2 & (1, \{[(2,0)]\}) & (1, \{[(2,1)]\}) & (0, \{\epsilon\}) & (2, \{[(2,1),(1,3)]\}) \\ 3 & \infty & \infty & \infty & (0, \{\epsilon\}) \end{array}$$

BGP-like solution — path-vectoring (4)

$\mathbf{B}_2$ and $\mathbf{B}_3$

$$\begin{array}{cc}
 & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \left[\begin{array}{cccc}
(0, \{\epsilon\}) & (1, \{[(0,1)]\}) & (1, \{[(0,2)]\}) & (3, \{[(0,2),(2,1),(1,3)]\}) \\
(1, \{[(1,0)]\}) & (0, \{\epsilon\}) & (1, \{[(1,2)]\}) & \infty \\
(1, \{[(2,0)]\}) & (1, \{[(2,1)]\}) & (0, \{\epsilon\}) & (3, \{[(2,0),(0,1),(1,3)]\}) \\
\infty & \infty & \infty & (0, \{\epsilon\})
\end{array}\right]
\end{array}$$

$$\begin{array}{cc}
 & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \left[\begin{array}{cccc}
(0, \{\epsilon\}) & (1, \{[(0,1)]\}) & (1, \{[(0,2)]\}) & \infty \\
(1, \{[(1,0)]\}) & (0, \{\epsilon\}) & (1, \{[(1,2)]\}) & \infty \\
(1, \{[(2,0)]\}) & (1, \{[(2,1)]\}) & (0, \{\epsilon\}) & \infty \\
\infty & \infty & \infty & (0, \{\epsilon\})
\end{array}\right]
\end{array}$$