L11 : Algebraic Path Problems with Applications to Internet Routing
Lectures 4 and 5

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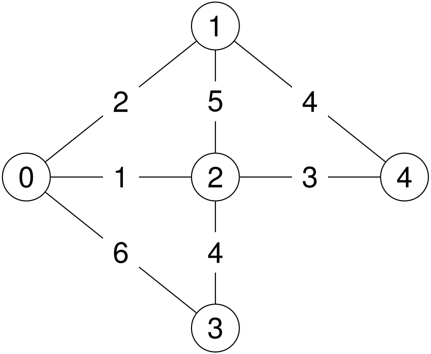
Shortest paths example, $(\mathbb{N}^{\infty}, \min, +)$



The adjacency matrix

$$\mathbf{A} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{ccccc} \infty & 2 & 1 & 6 & \infty \\ 2 & \infty & 5 & \infty & 4 \\ 1 & 5 & \infty & 4 & 3 \\ 6 & \infty & 4 & \infty & \infty \\ \infty & 4 & 3 & \infty & \infty \end{array}\right] \end{array}$$

OK OK, I changed node names ....

$(\min, +)$ example

Our theorem tells us that

$$\mathbf{A}^* = \mathbf{I} \ \min \ \mathbf{A} \ \min \ \mathbf{A}^2 \ \min \ \mathbf{A}^3 \ \min \ \mathbf{A}^4 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{ccccc} 0 & 2 & 1 & 5 & 4 \\ 2 & 0 & 3 & 7 & 4 \\ 1 & 3 & 0 & 4 & 3 \\ 5 & 7 & 4 & 0 & 7 \\ 4 & 4 & 3 & 7 & 0 \end{array} \right] \end{array}$$

$(\min, +)$ example

$$\mathbf{A}^2 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{ccccc} 2 & 6 & 7 & 5 & 4 \\ 6 & 4 & 3 & 8 & 8 \\ 7 & 3 & 2 & 7 & 9 \\ 5 & 8 & 7 & 8 & 7 \\ 4 & 8 & 9 & 7 & 6 \end{array}\right] \end{array}$$

$$\mathbf{A}^4 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{ccccc} 4 & 8 & 9 & 7 & 6 \\ 8 & 6 & 5 & 10 & 10 \\ 9 & 5 & 4 & 9 & 11 \\ 7 & 10 & 9 & 10 & 9 \\ 6 & 10 & 11 & 9 & 8 \end{array}\right] \end{array}$$

$$\mathbf{A}^3 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{ccccc} 8 & 4 & 3 & 8 & 10 \\ 4 & 8 & 7 & 7 & 6 \\ 3 & 7 & 8 & 6 & 5 \\ 8 & 7 & 6 & 11 & 10 \\ 10 & 6 & 5 & 10 & 12 \end{array}\right] \end{array}$$

Remember: we are looking at all paths of a given length, even those with cycles!

A “better” way — our basic algorithm

$$\begin{aligned} \mathbf{A}^{\langle 0 \rangle} &= \mathbf{I} \\ \mathbf{A}^{\langle k+1 \rangle} &= \mathbf{A}\mathbf{A}^{\langle k \rangle} \oplus \mathbf{I} \end{aligned}$$

Lemma

$$\mathbf{A}^{\langle k \rangle} = \mathbf{A}^{(k)} = \mathbf{I} \oplus \mathbf{A}^1 \oplus \mathbf{A}^2 \oplus \cdots \oplus \mathbf{A}^k$$

back to $(\min, +)$ example

$$\mathbf{A}^{\langle 1 \rangle} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{ccccc} 0 & 2 & 1 & 6 & \infty \\ 2 & 0 & 5 & \infty & 4 \\ 1 & 5 & 0 & 4 & 3 \\ 6 & \infty & 4 & 0 & \infty \\ \infty & 4 & 3 & \infty & 0 \end{array} \right] \end{array}$$

$$\mathbf{A}^{\langle 3 \rangle} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{ccccc} 0 & 2 & 1 & 5 & 4 \\ 2 & 0 & 3 & 7 & 4 \\ 1 & 3 & 0 & 4 & 3 \\ 5 & 7 & 4 & 0 & 7 \\ 4 & 4 & 3 & 7 & 0 \end{array} \right] \end{array}$$

$$\mathbf{A}^{\langle 2 \rangle} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{ccccc} 0 & 2 & 1 & 5 & 4 \\ 2 & 0 & 3 & 8 & 4 \\ 1 & 3 & 0 & 4 & 3 \\ 5 & 8 & 4 & 0 & 7 \\ 4 & 4 & 3 & 7 & 0 \end{array} \right] \end{array}$$

Let's build a "semiring of elementary paths"

Let $G = (V, E)$ be a finite directed graph. Define a sequence of arcs to be an elementary path from i to j

- the empty sequence is an elementary path from i to i
- $(i, j) \in E$ is an elementary path from i to j
- if $(i, u) \in E$ and p is an elementary path from u to j that does not contain i, then $(i, j), p$ is an elementary path from i to i.

Goal

A semiring S, such that if A is an adjaceny matrix over S with

$$A(i, j) = \begin{cases} \{(i, j)\} & \text{if } (i, j) \text{ is an arc} \\ \{\} & \text{otherwise} \end{cases}$$

then

$$A^*(i, j) = \text{the set of all elementary paths from } i \text{ to } j.$$

We could attempt to directly define such an algebra. But instead we will build it step-by-step using simple constructions ...

Lifted Product

Lifted product semigroup

If $S = (X, \otimes)$ be a semigroup, then let $\mathrm{lift}_{\times}(S) = (\mathcal{P}(X), \otimes_{\times})$ where

$$X \otimes_{\times} Y = \{x \otimes y \mid x \in X,\ y \in Y\}$$

Lifted product bi-semigroup

If $S = (X,\ \oplus,\ \otimes)$ be a bi-semigroup, then let

$$\mathrm{lift}_{\times}^{\cup}(S) = (\mathcal{P}(S),\ \cup,\ \otimes_{\times})$$

Reductions

If $(S, \oplus, \otimes)$ is bi-semigroup and r is a function from S to S, then r is a reduction if for all a and b in S

1. $r(a) = r(r(a))$
2. $r(a \oplus b) = r(r(a) \oplus b) = r(a \oplus r(b))$
3. $r(a \otimes b) = r(r(a) \otimes b) = r(a \otimes r(b))$

Note that if either operation has an identity, then the first axioms is not needed. For example,

$$r(a) = r(a \oplus \overline{0}) = r(r(a) \oplus \overline{0}) = r(r(a))$$

Reduce operation

If $S = (X, \oplus, \otimes)$ is bi-semigroup and r is a reduction, then let $\mathrm{red}_r(S) = (X_r, \oplus_r, \otimes_r)$ where

1. $X_r = \{x \in X \mid r(x) = x\}$
2. $x \oplus_r y = r(x \oplus y)$
3. $x \otimes_r y = r(x \otimes y)$

Is associativity preserved? Distributivity? Identities? Annihilators?

Finally : A semiring of elementary paths

Semigroup of Sequences $\mathrm{seq}(X)$

- carrier : finite sequences over elements of X
- operation : concatenation
- identity : the empty string ϵ

Let X be a set of sequences over E, and let

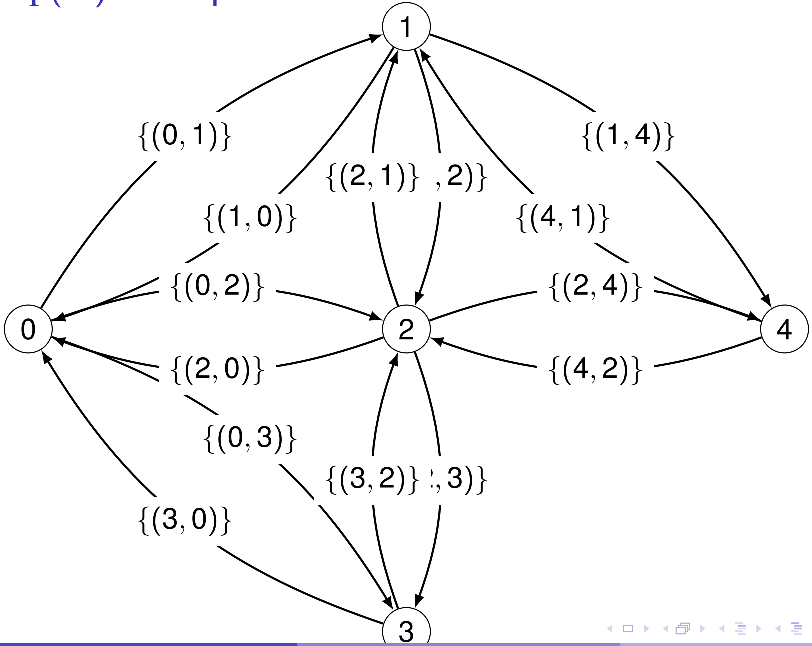
$$r(X) = \{p \in X \mid p \text{ is an elementary path in } G\}$$

Semiring of Elementary Paths

$$\mathrm{sep}(G) = \mathrm{red}_r(\mathrm{lift}_{\times}^{\cup}(\mathrm{seq}(E)))$$

In order to check that $\mathrm{sep}(G)$ is indeed a semiring, we only need understand the functions $\mathrm{lift}_{\times}^{\cup}(\_)$ and $\mathrm{red}\_(\_)$. (Left as an excersise.)

sep(G) example



sep(G) example, adjacency matrix

$$\mathbf{I} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{ccccc} \{\epsilon\} & \{\} & \{\} & \{\} & \{\} \\ \{\} & \{\epsilon\} & \{\} & \{\} & \{\} \\ \{\} & \{\} & \{\epsilon\} & \{\} & \{\} \\ \{\} & \{\} & \{\} & \{\epsilon\} & \{\} \\ \{\} & \{\} & \{\} & \{\} & \{\epsilon\} \end{array}\right] \end{array}$$

$$\mathbf{A} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{ccccc} \{\} & \{[(0,1)]\} & \{[(0,2)]\} & \{[(0,3)]\} & \{\} \\ \{[(1,0)]\} & \{\} & \{[(1,2)]\} & \{\} & \{[(1,4)]\} \\ \{[(2,0)]\} & \{[(2,1)]\} & \{\} & \{[(2,3)]\} & \{[(2,4)]\} \\ \{[(3,0)]\} & \{\} & \{[(3,2)]\} & \{\} & \{\} \\ \{\} & \{[(4,1)]\} & \{[(4,2)]\} & \{\} & \{\} \end{array}\right] \end{array}$$

Here I write a non-empty path p as $[p]$.

sep(G) example, solution

$$
\begin{aligned}
\mathbf{A}^*(0,0) &= \{\epsilon\} \\
\mathbf{A}^*(0,4) &= \left\{ \begin{array}{l}
[(0,1),(1,4)], \\
[(0,1),(1,2),(2,4)], \\
[(0,2),(2,4)], \\
[(0,2),(2,1),(1,4)], \\
[(0,3),(3,2),(2,4)], \\
[(0,3),(3,2),(2,1),(1,4)]
\end{array} \right\}
\end{aligned}
$$

Direct Product of Semigroups

Let $(S, \oplus_S)$ and $(T, \oplus_T)$ be semigroups.

Definition (Direct product semigroup)

The direct product is denoted $(S, \oplus_S) \times (T, \oplus_T) = (S \times T, \oplus)$, where $\oplus = \oplus_S \times \oplus_T$ is defined as

$$(s_1, t_1) \oplus (s_2, t_2) = (s_1 \oplus_S s_2,\ t_1 \oplus_T t_2).$$

Lexicographic Product of Semigroups

Definition (Lexicographic product semigroup)

Suppose that semigroup $(S, \oplus_S)$ is commutative, idempotent, and selective and that $(T, \oplus_T)$ is a semigroup. The lexicographic product is denoted $(S, \oplus_S) \vec{\times} (T, \oplus_T) = (S \times T, \vec{\oplus})$, where $\vec{\oplus} = \oplus_S \vec{\times} \oplus_T$ is defined as

$$(s_1, t_1) \vec{\oplus} (s_2, t_2) = \begin{cases} (s_1 \oplus_S s_2, t_1 \oplus_T t_2) & s_1 = s_1 \oplus_S s_2 = s_2 \\ (s_1 \oplus_S s_2, t_1) & s_1 = s_1 \oplus_S s_2 \neq s_2 \\ (s_1 \oplus_S s_2, t_2) & s_1 \neq s_1 \oplus_S s_2 = s_2 \end{cases}$$

Lexicographic product of Bi-semigroups

$$(S,\ \oplus_S,\ \otimes_S)\ \vec{\times}\ (T,\ \oplus_T,\ \otimes_T) = (S \times T,\ \oplus_S \vec{\times} \oplus_T,\ \otimes_S \times \otimes_T)$$

Theorem

If $\oplus_S$ is commutative, idempotent, and selective, then

$$\mathrm{LD}(S \vec{\times} T) \iff \mathrm{LD}(S) \wedge \mathrm{LD}(T) \wedge (\mathrm{LC}(S) \vee \mathrm{LK}(T))$$

Where

Property	Definition
LD	$\forall a, b, c : c \otimes (a \oplus b) = (c \otimes a) \oplus (c \otimes b)$
LC	$\forall a, b, c : c \otimes a = c \otimes b \implies a = b$
LK	$\forall a, b, c : c \otimes a = c \otimes b$

Prove

$\text{LD}(S) \wedge \text{LD}(T) \wedge (\text{LC}(S) \vee \text{LK}(T)) \implies \text{LD}(S \vec{\times} T)$

Assume S and T are bisemigroups, $\text{LD}(S) \wedge \text{LD}(T) \wedge (\text{LC}(S) \vee \text{LK}(T))$, and

$$(s_1, t_1), (s_2, t_2), (s_3, t_3) \in S \times T.$$

Then (dropping operator subscripts for clarity) we have

$$\begin{aligned}
\text{lhs} &= (s_1, t_1) \otimes ((s_2, t_2) \vec{\oplus} (s_3, t_3)) \\
&= (s_1, t_1) \otimes (s_2 \oplus s_3, t_{\text{lhs}}) \\
&= (s_1 \otimes (s_2 \oplus s_3), t_1 \otimes t_{\text{lhs}})
\end{aligned}$$

$$\begin{aligned}
\text{rhs} &= ((s_1, t_1) \otimes (s_2, t_2)) \vec{\oplus} ((s_1, t_1) \otimes (s_3, t_3)) \\
&= (s_1 \otimes s_2, t_1 \otimes t_2) \vec{\oplus} (s_1 \otimes s_3, t_1 \otimes t_3) \\
&= ((s_1 \otimes s_2) \oplus_S (s_1 \otimes s_3), t_{\text{rhs}}) \\
&= (s_1 \otimes (s_2 \oplus s_3), t_{\text{rhs}})
\end{aligned}$$

where t_{lhs} and t_{rhs} are determined by the definition of $\vec{\oplus}$.

We need to show that $lhs = rhs$, that is $t_{\text{rhs}} = t_1 \otimes t_{\text{lhs}}$.

Case 1 : $\text{LC}(S)$

Note that we have

$$(\star) \quad \forall a, b, c : a \neq b \implies c \otimes a \neq c \otimes b$$

Case 1.1 : $s_2 = s_2 \oplus s_3 = s_3$. Then $t_{\text{lhs}} = t_2 \oplus t_3$ and $t_1 \otimes t_{\text{lhs}} = t_1 \otimes (t_2 \oplus t_3) = (t_1 \otimes t_2) \oplus (t_1 \otimes t_3)$, by $\text{LD}(S)$. Also, $s_1 \otimes_S s_2 = s_1 \otimes_S s_3$ and $s_1 \otimes s_2 = s_1 \otimes (s_2 \oplus s_3) = (s_1 \otimes s_2) \oplus (s_1 \otimes s_3)$, again by $\text{LD}(S)$. Therefore $t_{\text{rhs}} = (t_1 \otimes t_2) \oplus (t_1 \otimes t_3) = t_1 \otimes t_{\text{lhs}}$.

Case 1.2 : $s_2 = s_2 \oplus s_3 \neq s_3$. Then $t_1 \otimes t_{\text{lhs}} = t_1 \otimes t_2$ Also $s_2 = s_2 \oplus s_3 \implies s_1 \otimes s_2 = s_1 \otimes (s_2 \oplus s_3)$ and by $\star$ $s_2 \oplus s_3 \neq s_3 \implies s_1 \otimes (s_2 \oplus s_3) \neq s_1 \otimes s_3$. Thus, by $\text{LD}(S)$, $(s_1 \otimes s_2) \oplus (s_1 \otimes s_3) \neq s_1 \otimes s_3$ and we get $t_{\text{rhs}} = t_1 \otimes t_2 = t_1 \otimes t_{\text{lhs}}$.

Case 1.3 : $s_2 \neq s_2 \oplus_S s_3 = s_3$. Similar to case 1.2.

Case 2 : $\text{LK}(T)$

Case 2.1 : $s_2 = s_2 \oplus_S s_3 = s_3$. Same as Case 1.1.

Case 2.2 : $s_2 = s_2 \oplus_S s_3 \neq s_3$. Then $t_1 \otimes t_{\text{lhs}} = t_1 \otimes t_2$. Now, $(s_1 \otimes s_2) \oplus_S (s_1 \otimes s_3) = s_1 \otimes (s_2 \oplus s_3) = s_1 \otimes s_2$. So $t_{\text{rhs}} = (t_1 \otimes t_2) \oplus (t_1 \otimes t_3) = t_1 \otimes (t_2 \oplus t_3)$ or $t_{\text{rhs}} = (t_1 \otimes t_2)$. In either case, t_{rhs} is of the form $t_1 \otimes t$, so by $\text{LK}(T)$ we know that $t_{\text{rhs}} = t_1 \otimes t_{\text{lhs}}$.

Case 2.3 : $s_2 \neq s_2 \oplus_S s_3 = s_3$. Similar to case 2.2.

Examples

name	LD	LC	LK
min_plus	Yes	Yes	No
max_min	Yes	No	No
$\mathrm{sep}(G)$	Yes	No	No

So we have

$$\mathsf{LD}(\mathrm{min\_plus} \vec{\times} \mathrm{max\_min})$$
$$\mathsf{LD}(\mathrm{min\_plus} \vec{\times} \mathrm{sep}(G))$$

But

$$\neg(\mathsf{LD}(\mathrm{max\_min} \vec{\times} \mathrm{min\_plus}))$$
$$\neg(\mathsf{LD}(\mathrm{sep}(G) \vec{\times} \mathrm{min\_plus}))$$

Shortest paths with best paths

Let's use

$$\text{add\_zero}(\infty,\ \text{min\_plus} \vec{\times} \text{sep}(G))$$

$$\mathbf{I} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{ccccc} (0, \{\epsilon\}) & \infty & \infty & \infty & \infty \\ \infty & (0, \{\epsilon\}) & \infty & \infty & \infty \\ \infty & \infty & (0, \{\epsilon\}) & \infty & \infty \\ \infty & \infty & \infty & (0, \{\epsilon\}) & \infty \\ \infty & \infty & \infty & \infty & (0, \{\epsilon\}) \end{array}\right] \end{array}$$

$$\mathbf{A} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & \end{array} \\ \left[\begin{array}{ccccc} \infty & (2, \{[(0,1)]\}) & (1, \{[(0,2)]\}) & (6, \{[(0,3)]\}) & \\ (2, \{[(1,0)]\}) & \infty & (5, \{[(1,2)]\}) & \infty & (4, \{[\\ (1, \{[(2,0)]\}) & (5, \{[(2,1)]\}) & \infty & (4, \{[(2,3)]\}) & (3, \{[\\ (6, \{[(3,0)]\}) & \infty & (4, \{[(3,2)]\}) & \infty & \\ \infty & (4, \{[(4,1)]\}) & (3, \{[(4,2)]\}) & \infty & \end{array}\right. \end{array}$$

Solution

$$\mathbf{A}^* = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{ccc} 0 & 1 & 2 \\ \left[\begin{array}{c} (0, \{\epsilon\}) \\ (2, \{[(1,0)]\}) \\ (1, \{[(2,0)]\}) \\ (5, \{[(3,2),(2,0)]\}) \\ (4, \{[(4,2),(2,0)]\}) \end{array}\right. & \begin{array}{c} (2, \{[(0,1)]\}) \\ (0, \{\epsilon\}) \\ (3, \{[(2,0),(0,1)]\}) \\ (7, \{[(3,2),(2,0),(0,1)]\}) \\ (4, \{[(4,1)]\}) \end{array} & \begin{array}{c} (1, \{[(0,2)]\}) \\ (3, \{[(1,0),(0,2)]\}) \\ (0, \{\epsilon\}) \\ (4, \{[(3,2)]\}) \\ (3, \{[(4,2)]\}) \end{array} \end{array}$$