L11 : Algebraic Path Problems with Applications to Internet Routing

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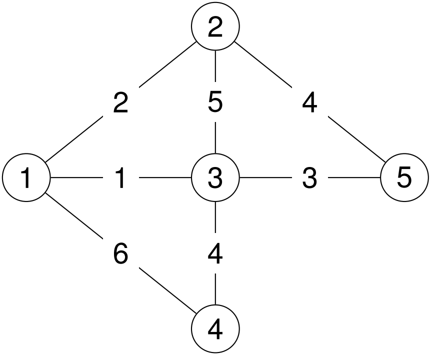
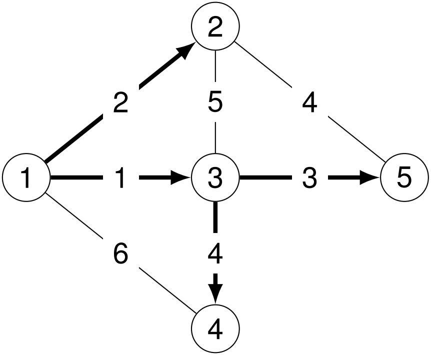
Shortest paths example, $(\mathbb{N}^\infty, \min, +)$



The adjacency matrix

$$\mathbf{A} = \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & \infty & 2 & 1 & 6 & \infty \\ 2 & 2 & \infty & 5 & \infty & 4 \\ 3 & 1 & 5 & \infty & 4 & 3 \\ 4 & 6 & \infty & 4 & \infty & \infty \\ 5 & \infty & 4 & 3 & \infty & \infty \end{array}$$

Shortest paths example, $(\mathbb{N}^{\infty}, \min, +)$



Bold arrows indicate the shortest-path tree rooted at 1.

The routing matrix

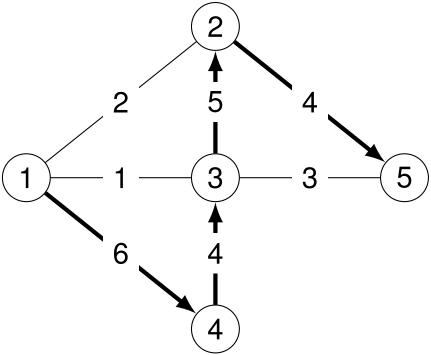
$$\mathbf{R} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \left[\begin{array}{ccccc} 0 & 2 & 1 & 5 & 4 \\ 2 & 0 & 3 & 7 & 4 \\ 1 & 3 & 0 & 4 & 3 \\ 5 & 7 & 4 & 0 & 7 \\ 4 & 4 & 3 & 7 & 0 \end{array} \right] \end{array}$$

Matrix **R** solves this global optimality problem:

$$\mathbf{R}(i, j) = \min_{p \in P(i, j)} w(p),$$

where $P(i, j)$ is the set of all paths from i to j.

Widest paths example, $(\mathbb{N}^\infty, \max, \min)$



Bold arrows indicate the widest-path tree rooted at 1.

The routing matrix

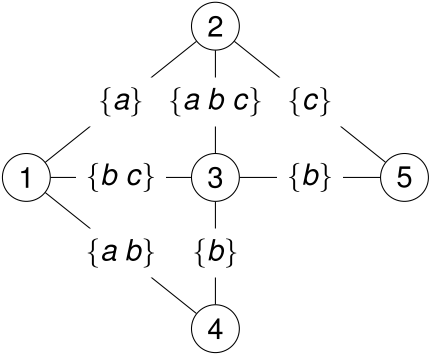
$$\mathbf{R} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \left[\begin{array}{ccccc} \infty & 4 & 4 & 6 & 4 \\ 4 & \infty & 5 & 4 & 4 \\ 4 & 5 & \infty & 4 & 4 \\ 6 & 4 & 4 & \infty & 4 \\ 4 & 4 & 4 & 4 & \infty \end{array}\right] \end{array}$$

Matrix **R** solves this global optimality problem:

$$\mathbf{R}(i, j) = \max_{p \in P(i, j)} w(p),$$

where $w(p)$ is now the minimal edge weight in p.

Unfamiliar example, $(2^{\{a,\ b,\ c\}},\ \cup,\ \cap)$



We want a Matrix **R** to solve this global optimality problem:

$$\mathbf{R}(i,\ j) = \bigcup_{p \in P(i,\ j)} w(p),$$

where $w(p)$ is now the intersection of all edge weights in p.

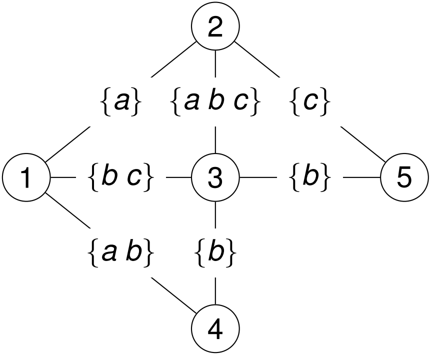
For $x \in \{a,\ b,\ c\}$, interpret $x \in \mathbf{R}(i,\ j)$ to mean that there is at least one path from i to j with x in every arc weight along the path.

Unfamiliar example, $(2^{\{a,\ b,\ c\}},\ \cup,\ \cap)$

The matrix **R**

$$
\begin{array}{c|ccccc}
 & 1 & 2 & 3 & 4 & 5 \\
\hline
1 & \{a\ b\ c\} & \{a\ b\ c\} & \{a\ b\ c\} & \{a\ b\} & \{b\ c\} \\
2 & \{a\ b\ c\} & \{a\ b\ c\} & \{a\ b\ c\} & \{a\ b\} & \{b\ c\} \\
3 & \{a\ b\ c\} & \{a\ b\ c\} & \{a\ b\ c\} & \{a\ b\} & \{b\ c\} \\
4 & \{a\ b\} & \{a\ b\} & \{a\ b\} & \{a\ b\ c\} & \{b\} \\
5 & \{b\ c\} & \{b\ c\} & \{b\ c\} & \{b\} & \{a\ b\ c\}
\end{array}
$$

Another unfamiliar example, $(2^{\{a,\ b,\ c\}},\ \cap,\ \cup)$



We want matrix **R** to solve this global optimality problem:

$$\mathbf{R}(i,\ j) = \bigcap_{p \in P(i,\ j)} w(p),$$

where $w(p)$ is now the union of all edge weights in p.

For $x \in \{a,\ b,\ c\}$, interpret $x \in \mathbf{R}(i,\ j)$ to mean that every path from i to j has at least one arc with weight containing x.

Another unfamiliar example, $(2^{\{a,\ b,\ c\}},\ \cap,\ \cup)$

The matrix **R**

$$
\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}
\begin{array}{c}
\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\
\left[\begin{array}{ccccc}
\{\} & \{\} & \{b\} & \{b\} & \{\} \\
\{\} & \{\} & \{b\} & \{b\} & \{\} \\
\{b\} & \{b\} & \{\} & \{b\} & \{b\} \\
\{b\} & \{b\} & \{b\} & \{\} & \{b\} \\
\{\} & \{\} & \{b\} & \{b\} & \{\}
\end{array}\right]
\end{array}
$$

What this course is about ...

- How can we capture “generalized path problems”?
- How can we compute solutions?
- How do familiar algorithms (Dijkstra, Bellman-Ford, Floyd-Warshall, etc) fit into the picture?
- Can we use this knowledge to “reverse engineer” Internet routing protocols?
- Can we use this knowledge to design new routing protocols?
- Can we use this knowledge in other application areas?

We will start by looking at Semirings

name	S	$\oplus$,	$\otimes$	$\overline{0}$	$\overline{1}$	possible routing use
sp	$\mathbb{N}^{\infty}$	min	$+$	∞	0	minimum-weight routing
bw	$\mathbb{N}^{\infty}$	max	min	0	∞	greatest-capacity routing
rel	$[0, 1]$	max	$\times$	0	1	most-reliable routing
use	$\{0, 1\}$	max	min	0	1	usable-path routing
	2^W	$\cup$	$\cap$	$\{\}$	W	shared link attributes?
	2^W	$\cap$	$\cup$	W	$\{\}$	shared path attributes?

A wee bit of notation!

Symbol	Interpretation
$\mathbb{N}$	Natural numbers (starting with zero)
$\mathbb{N}^{\infty}$	Natural numbers, plus infinity
$\overline{0}$	Identity for $\oplus$
$\overline{1}$	Identity for $\otimes$

The (Tentative) Plan

1	5 October	:	Motivation, overveiw
2	9 October	:	Semigroups and Orders
3	12 October	:	Semirings — Theory
4	16 October	:	Semirings — Constructions
5	19 October	:	Semirings — Examples
6	23 October	:	Semirings — algorithms
7	26 October	:	Beyond Semirings — "functions on arcs"
8	30 October	:	Path finding vs Routing (**HW 1 due**)
9	2 November	:	Graph (Network) decomposition
10	6 November	:	Protocols : RIP, OSPF, IS-IS
11	9 November	:	Beyond Semirings — Global vs Local optimality
12	3 November	:	More on Global vs Local optimality
13	16 November	:	Protocols : EIGRP, BGP
14	20 November	:	Dijkstra revisited
15	23 November	:	Route redistribution, administrative distance
16	27 November	:	Metarouting project (**HW 2 due**)
	15 January	:	**HW 3 due**

Reading related to topic of next 5 lectures

J. Inst. Maths Applics (1971) **7**, 273–294

An Algebra for Network Routing Problems

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