Countable Sets

The prototypical infinite countable sets: $\mathbb{N}_0$, $\mathbb{N}$.

For a set S, the following are equivalent:

1. S is finite or there is a bijection $\mathbb{N} \to S$.
2. There is a bijection $A \to S$ for $A \subseteq \mathbb{N}$.
3. There is an injection $S \to \mathbb{N}$.
4. S is empty or there is a surjection $\mathbb{N} \to S$.

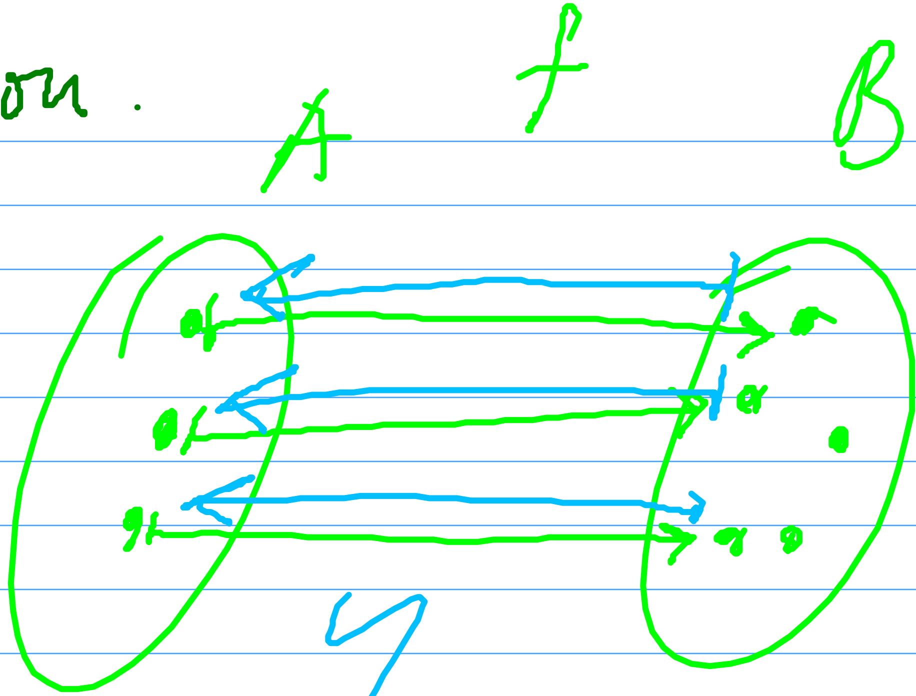
Countable sets are those for which the above hold.

For an infinite set S, the following are equivalent:

1. There is a bijection $\mathbb{N} \to S$
3. There is an injection $S \to \mathbb{N}$
4. There is a surjection $\mathbb{N} \to S$

$(3) \Rightarrow (4)$ <u>Key Lemma</u> For every injection $f: A \to B$, the inverse relation $f^{-1} \subseteq B \times A$ (given by $b \, f^{-1} \, a$ iff$_{\text{def}}$ $f(a) = b$) is a

partial surjective function.



is a partial function that is moreover surjective

Assume there is an injection

$$S \xrightarrow[f]{} \mathbb{N}$$

Consider the partial surjective function

$$f^{-1} : \mathbb{N} \to S$$

and define

$$g : \mathbb{N} \to S \text{ given by } g(n) = \begin{cases} f^{-1}(n) & n \in \text{dom} \\ s & \text{otw} \end{cases}$$

for some $s \in S$

(4) ⇒ (1) If there is a surjection $\mathbb{N} \xrightarrow{e} S$

then there is a bijection $\mathbb{N} \xrightarrow{f} S$.

$e \qquad e(0), e(1), \ldots, e(n), \ldots$

$f \qquad e(0), \begin{cases} \text{if } e(1) \neq e(0) \text{ then } e(1) \\ \text{try the next one.} \end{cases}$

NB ⟹ We have introduced a new operation

Formally, define $f(0) = e(0)$.

$$f(n+1) = e\left(\min\{k \text{ s.t. } e(k) \neq f(0), f(1), \ldots, f(n)\}\right)$$

minimisation.

↳ important in recursion theory

$\mathcal{P}_{fin}(\mathbb{N})$ is countable

$\mathbb{N} \cong \mathbb{N}_0$ ($\xrightarrow{-1}$, $\xleftarrow{+1}$)

$\mathbb{N} \xrightarrow{f} \mathbb{Z}$: $f(n) = \begin{cases} k & n = 2k \\ -k & n = 2k+1 \end{cases}$

Examples:

- $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$.
- $\mathbb{N}_0 \times \mathbb{N}_0$, $\mathbb{Q}$.
- $A \times B$ for A and B countable.
- A^n for A countable and $n \in \mathbb{N}_0$.

$\{(n,m) \mid n \in \mathbb{N}_0, m \in \mathbb{N}_0\}$

Produce an enumeration

$\mathbb{N} \to \mathbb{N}_0 \times \mathbb{N}_0$

surj enumeration

$\mathbb{N} \to \mathbb{Z} \times \mathbb{N} \longrightarrow \mathbb{Q}$

$p, q \longmapsto p/q$

$\left.\begin{array}{l} \mathbb{N} \xrightarrow{e_1} A \\ \mathbb{N} \xrightarrow{e_2} B \end{array}\right\}$

surj enum.

$\mathbb{N} \to \mathbb{N} \times \mathbb{N} \longrightarrow A \times B$

$(n, m) \longmapsto (e_1 n, e_2 m)$

$j \quad (i,j) \in \mathbb{N}_0 \times \mathbb{N}_0$

$i \backslash j$	0	1	2	3	4	...	
0	0	1	3	6	10	15 –	21
1	2	4	7	11	16	22	
2	5	8	12	17	23		
3	9	13	18				
4	14	19					
⋮	20						

$23 = [1+2+ \cdots + 6] + 2$

$i \qquad i+j = 2+4$

Define

$$\mathbb{N}_0 \times \mathbb{N}_0 \xrightarrow{\text{bij}} \mathbb{N}_0$$

$$(i,j) \longmapsto \frac{(i+j)(i+j+1)}{2} + i$$

The set $(\mathbb{N} \Rightarrow [2])$ is uncountable! $\cong \mathcal{P}(\mathbb{N})$

Uncountable Sets

– diagonalisation –

NOTES

$(0,1)$ is uncountable

① Basic idea is to think of function $f: \mathbb{N} \to [2]$ as infinite bit streams.

$$f: \quad f(0), f(1), f(2), \ldots, f(n), \ldots \qquad (n \in \mathbb{N})$$

② Proceed by contradiction:

Assume the set $(\mathbb{N} \Rightarrow [2])$ "of infinite bit streams" is countable.

So there is an enumeration of them say

$$e : \mathbb{N} \xrightarrow[\text{surj}]{} (\mathbb{N} \Rightarrow [2])$$

Pictorially

e_0 : $e_0(0)$, $e_0(1)$, $e_0(2)$, - - - . . , $e_0(n)$, - . . .

e_1 : $e_1(0)$, $e_1(1)$, $e_1(2)$, - - - . . , $e_1(n)$, - - .

$\vdots$

notation:

$\overline{0} = 1$

$\overline{1} = 0$

e_n : $e_n(0)$, $e_n(1)$, - . , $e_n(n)$, - - .

Claim: There is a bit stream β that is not in this enumeration. That is $\beta \neq e_n$ for all n

β : $\overline{e_0(0)}$, $\overline{e_1(1)}$, $\overline{e_2(2)}$, - . . , $\overline{e_n(n)}$, - - -

Define $\beta(n) = \overline{e_n(n)} \implies \beta \neq e_n$

because β and e_n differ in the n^{th} bit by construction of β.

Hence e is not a surjective enumeration.

A contradiction! $\boxtimes$

Corollary: There are non-computable infinite bit streams.

Chapter 4

Reading list:

4.1 Russell's paradox

4.2 Constructing sets

4.3 Some consequences

Suggested exercises: 4.4, 4.5, 4.6, 4.7, 4.8, 4.9

Russell's Paradox

The construction $\{\, x \mid x \notin x \,\}$ should not yield a set!

Because it leads to contradiction!

Constructions on Sets

Basic sets:

- $\{0, 1\}$
- $\{a, b, c, \ldots, x, y, z\}$
- $\mathbb{N}_0 =_{\text{def}} \{0, 1, 2, \ldots\}$

Comprehension:

For every set X and property $P(x)$ for x ranging over X, we can form the set

$$\{\, x \in X \mid P(x) \,\}$$

$\rightsquigarrow$ Rules out Russell's paradox.

Example:

$$\{\, 2m + 1 \mid m \in \mathbb{N} \text{ and } m > 1 \,\}$$

formally

$$\{ k \in \mathbb{N} \mid \exists m \in \mathbb{N}.\ m > 1 \ \& \ k = 2m+1 \}$$

Powerset:

For every set X, we can form the set

$$\mathcal{P}(X) =_{\mathrm{def}} \{\, S \mid S \subseteq X\}$$

consisting of all the subsets of X.