BIJECTIONS

$f: A \to B$ is bijection whenever it has an inverse

i.e. a function $g: B \to A$ s.t. $g \circ f = id_A$ & $f \circ g = id_B$.

NB: If f has an inverse then it is unique. For if g and h are both inverses of f then

$$g = g \circ id = g \circ f \circ h = id \circ h = h$$

The notation for the inverse of f, whenever it exists, is f^{-1}

If $\#A = n$ and $\#B = m$,

$$\#\mathrm{Bij}(A,B) = \begin{cases} 0 & n \neq m \\ n! & n = m \end{cases}$$

The set of all bijections from A to B?

$a_1 \longmapsto$ $b_1 \quad b_2 \quad b_j \quad b_m$ $\qquad m$

$a_2 \longmapsto$ $b_1 \ldots \quad b_m$ — but not b_j $\qquad (m-1)$

$\vdots$ $\qquad (m-2)$

$a_n \longmapsto$ $\qquad \vdots$

Bijections are also called permutations or isomorphisms.

$$\begin{tikzcd} \underline{\mathrm{EqRel}}(A) \arrow[rr, bend left, "\pi"] \arrow[rr, phantom, "\cong"] & & \underline{\mathrm{Part}}(A) \arrow[ll, bend left, "\varepsilon"] \end{tikzcd}$$

notation for bijections

$$\pi(R) = \{ [a]_R \mid a \in A \}$$

where $[a]_R = \{ x \in A \mid x R a \}$

typically denoted A/R — The quotient of A under R

$A = \mathbb{Z} \times \mathbb{N}$ define $(p,q) \equiv (p',q')$ — is an equivalence relation

iff $p \times q' = p' \times q$

Consider $(\mathbb{Z}\times\mathbb{N})/_{\equiv} \underset{\text{Exercise.}}{\cong} \mathbb{Q}$

$$\mathbb{Q} \to (\mathbb{Z}\times\mathbb{N})/_{\equiv} \qquad (\mathbb{Z}\times\mathbb{N})/_{\equiv} \longrightarrow \mathbb{Q}$$

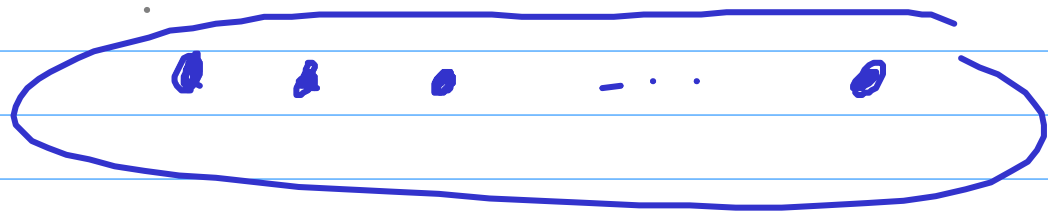
$$p/q \underset{\text{def}}{\longmapsto} [(p,q)]_{\equiv} \qquad [(p,q)]_{\equiv} \longmapsto p/q$$

$\# A = n$ $\qquad$ $\#\underline{Part}(A) = \sum_k S(n,k)$

The number of partitions of an n element set into k blocks.

$$S(n,0) = \begin{cases} 0 & n \neq 0 \\ 1 & n = 0 \end{cases}$$

$$S(n,1) = 1$$



Stirling number of the second kind.

$S(n+1, k+1)$

$= S(n,k) + S(n,k+1) \times (k+1)$

$1 \quad 2 \quad 3 \quad \cdots \quad n \quad n+1$

Case 1

n+1 is in a block on its own

$1 \quad 2 \quad 3 \quad \ldots \quad n \quad n+1$

$S(n,k)$

Case 2

n+1 is not in a block of its own.

$1 \quad 2 \quad 3 \quad \cdots \quad n$

$S(n, k+1) \times (k+1)$

Terminology: The image of f

Surjective functions

$f : A \to B$ bijective

$f(A) \overset{def}{=} \{ f(a) \mid a \in A \} \subseteq B$

? What can we say about the codomain of a bijective function?

For f bijective, $f(A) = B$

Def $f : A \to B$ is surjective if the following equivalent conditions hold:

- $f(A) = B$
- $\forall b \in B \,.\, \exists a \in A \,.\, f(a) = b$.

Examples: ...

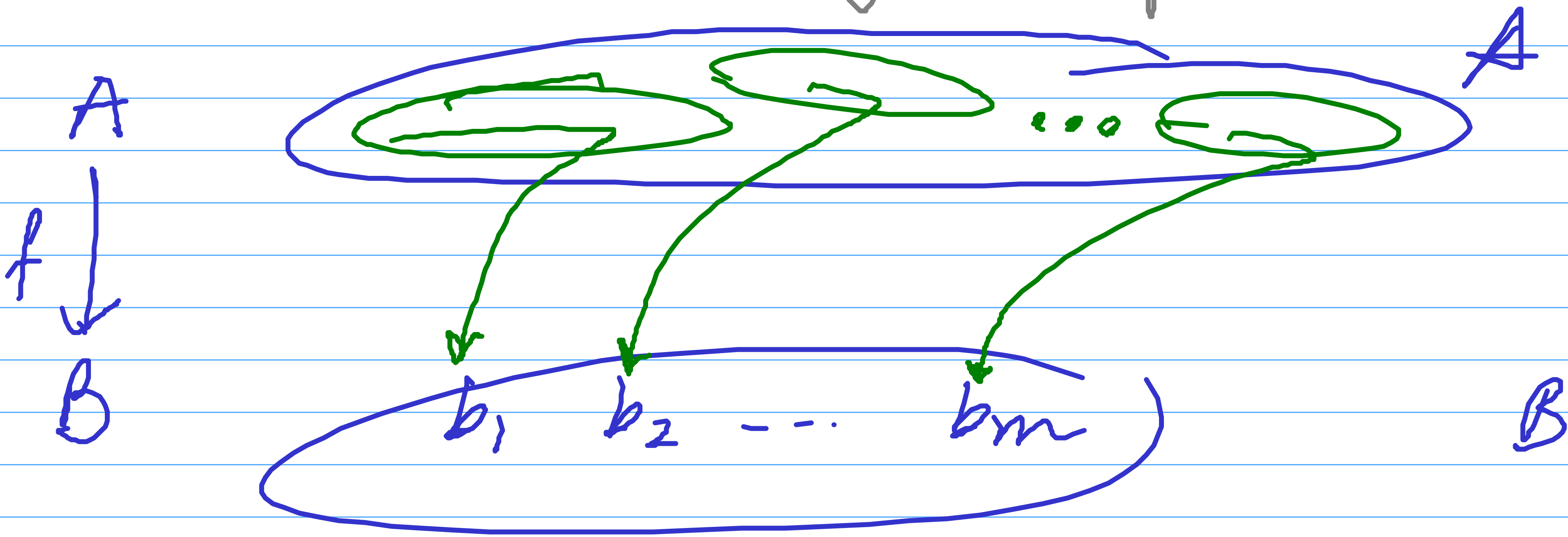
Let R be an equivalence relation on A. Then the quotient function $q : A \to A/R$ defined as $q(a) = [a]_R$ is a surjection

Let $\underline{Sur}(A,B)$ be the set of surjection from A to B.

$\# A = n$, $\# B = m$ $\qquad \# \underline{Sur}(A,B) = \quad ?$

Let $f : A \to B$ be surjective

equivalently an ordered partition



$\# \underline{Sur}(A,B) = S(n,m) \times m!$

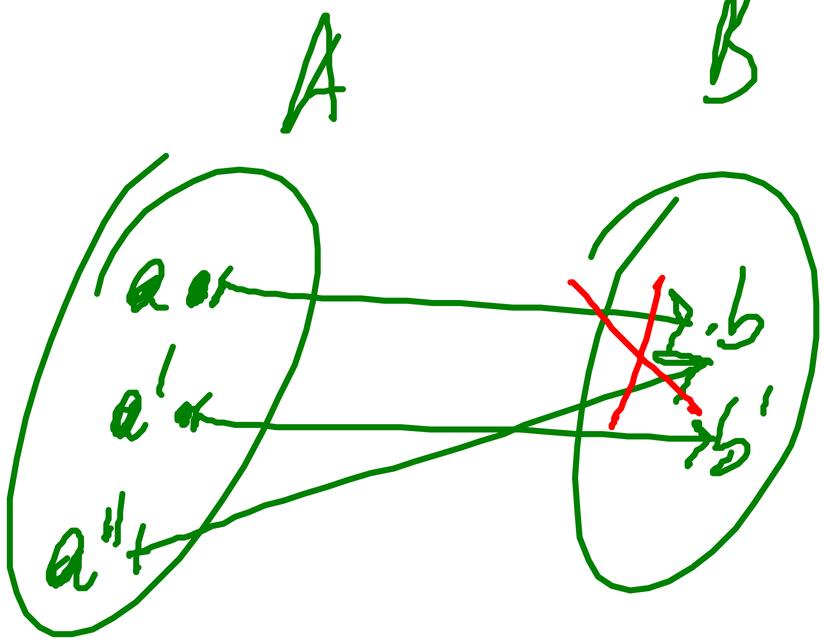
when no two inputs produce the same output

Injective functions

Def $f: A \to B$ is injective

iff (def) $\forall a, a' \in A.\ f(a) = f(a') \Rightarrow a = a'$.

iff $\forall a \neq a'$ in A. $f(a) \neq f(a')$.



Examples: ...

(1) Every bijection is an injection

(2) Every inclusion is an injection: for $A \subseteq B$ the function $i: A \longrightarrow B$ given by $i(a) = a$ is an injection.

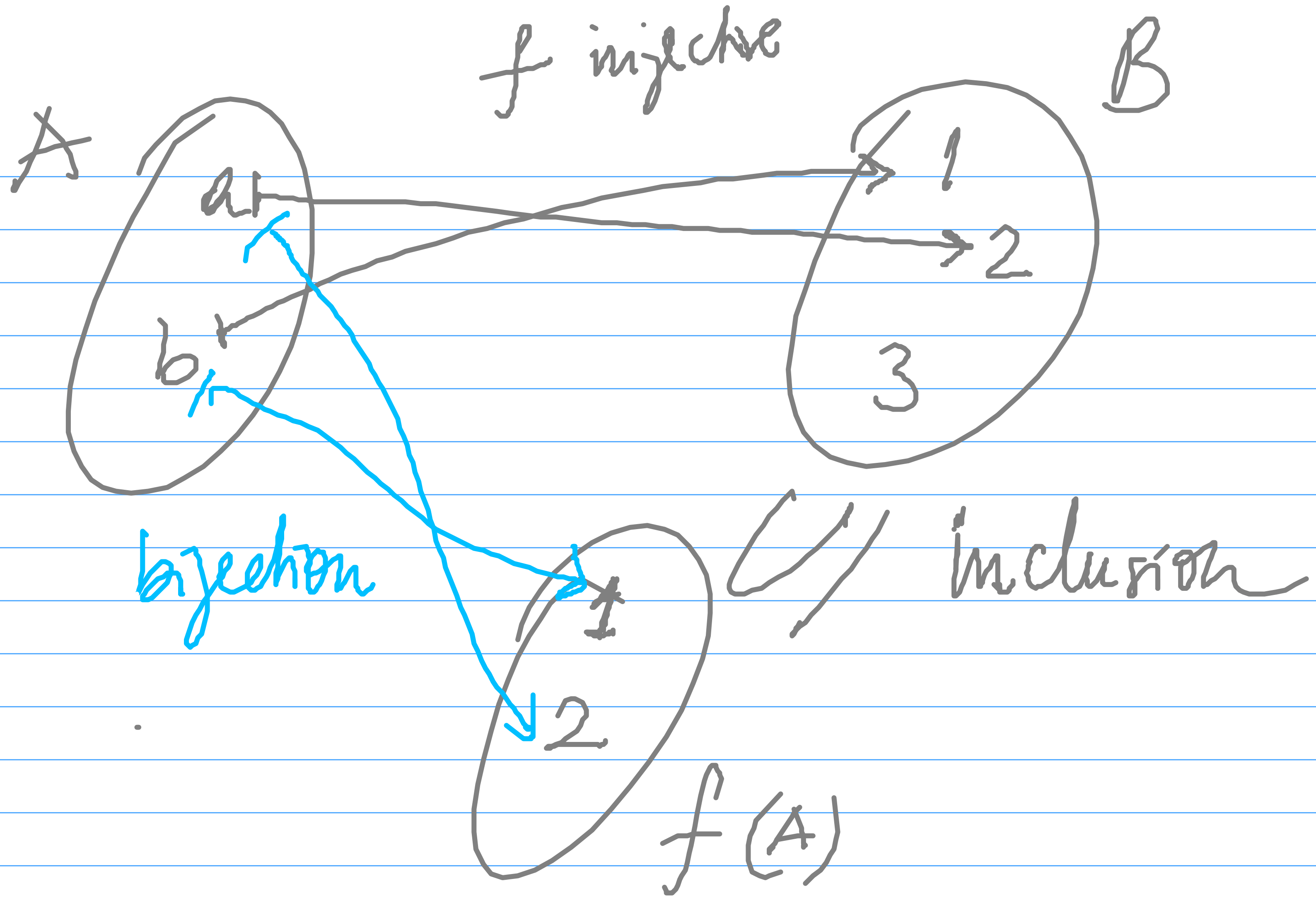
Claim: Every injective function $f: A \to B$ decomposes as the composition of a bijection followed by an inclusion.

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\text{bijection}"'] & & B \\ & I \arrow[ur, hook, "\subseteq"'] & \end{tikzcd}$$

Consider I to be the image of f; that is

$$I = f(A) = \{ f(a) \mid a \in A \} \subseteq B$$

Example



Def:

$f' : A \longrightarrow f(A)$

$f'(a) \stackrel{def}{=} f(a)$

This is bijective because f is injective

And

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "f'"'] & & B \\ & f(A) \arrow[ur, hook, "\text{inclusion}"'] & \end{tikzcd}$$

bijection

$\#A = n$

$\#B = m$

$$\# \mathrm{Inj}(A, B) = n! \cdot \binom{m}{n}$$

$$= m \times (m-1) \times \cdots \times (m-n+1)$$

notation

$$= m^{\underline{n}}$$

The n^{th} FALLING POWER of m