(Total) Functions

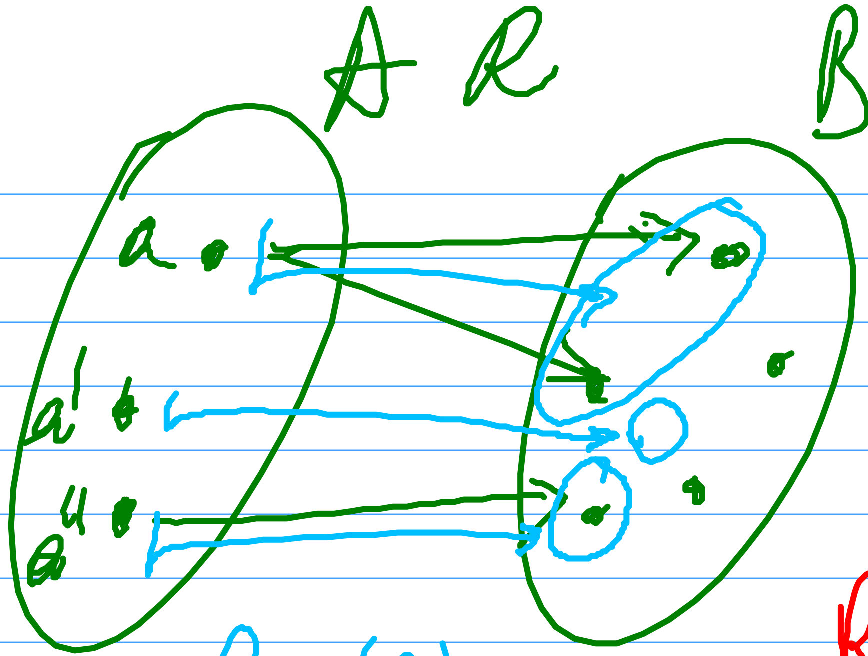
- A *function* is a partial function defined on every element of the domain; that is, . . .

<u>Def</u> A <u>function</u> f from A to B (notation $f : A \to B$) is a partial function $f : A \to B$ such that $\underline{\text{dom}}(f) = A$

Examples: . . . Every relation R from A to B induces a function from A to $\mathcal{P}(B)$

$R \subseteq A \times B$

$\underline{fun}(R) : A \to \mathcal{P}(B)$



$\underline{fun}(R)$

$\forall a \in A.$

$\underline{fun}(R)(a) \overset{def}{=} \{b \in B \mid aRb\} \in \mathcal{P}(B)$

$\underline{rel}(\underline{fun}(R)) \overset{?}{=} R$

$\underline{fun}(\underline{rel}(f)) \overset{?}{=} f$

Given $f : A \to \mathcal{P}(B)$

$\underline{rel}(f) \subseteq A \times B$

def: $\forall a \in A, b \in B,\ (a,b) \in \underline{rel}(f)$ iff$_{def}$ $b \in f(a)$

$$\begin{tikzcd} \mathcal{P}(A \times B) \arrow[rr, bend left, "\underline{fun}"] & & (A \Rightarrow \mathcal{P} B) \arrow[ll, bend left, "\underline{rel}"] \end{tikzcd}$$

$\forall R.\quad \underline{rel}\,(\underline{fun}\ R) = R$

$\forall f.\quad \underline{fun}\,(\underline{rel}\ f) = f$

Bijective correspondence

Equivalently

$$\underline{rel} \circ \underline{fun} = id_{\mathcal{P}(A \times B)}$$

$$\underline{fun} \circ \underline{rel} = id_{(A \Rightarrow \mathcal{P} B)}$$

$\boxed{?}$ Is the relational composition of two functions a function?

$f : A \to B$ $g : B \to C$

$g \circ f \in Rel(A, C)$

$(g \circ f)(a) = g(f(a))$ $\forall a \in A$

Yes!

$k^0 = 1 \rightsquigarrow \#(\emptyset \Rightarrow k) = 1$

$\boxed{?}$ Is the relational composition of two functions a function?

<u>Notation</u> $B^A \rightsquigarrow$ The set $(A \Rightarrow B)$

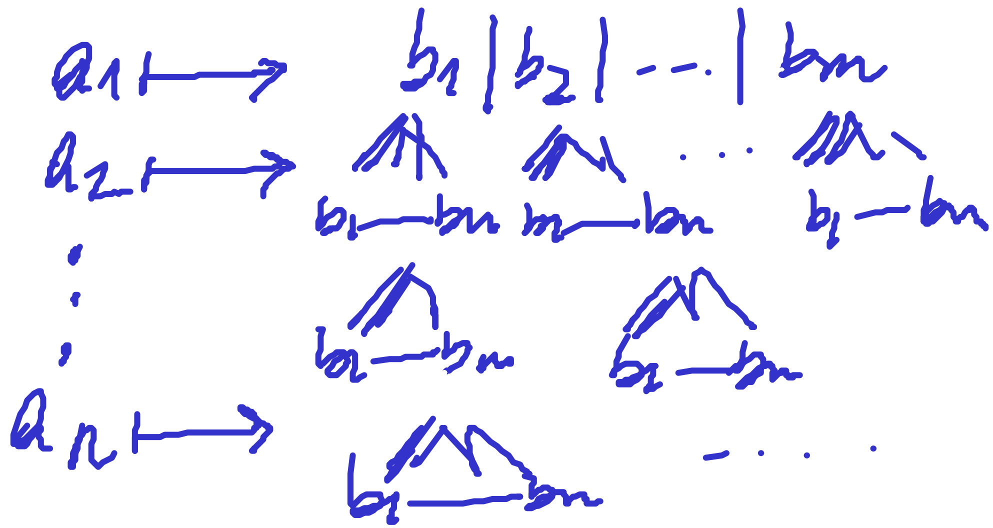
$\boxed{?}$ What is the cardinality of the set of functions between two finite sets?

$\#A = n \qquad \#B = m$

$\#(A \Rightarrow B)$

$\|$

m^n



$$\left.\begin{array}{l} m \\ \times\ m \\ \times\ \vdots \\ \times\ m \end{array}\right\} n \text{ times}$$

Bijective functions

or bijections

$Bij(A,B) \subseteq (A \Rightarrow B)$

def: $f : A \to B$ is a bijection iff

① $\exists g : B \to A$ such that $g \circ f = id_A$ and $f \circ g = id_B$

② $\forall b \in B.\ \exists!\, a \in A.\ f(a) = b$.

③ f is surjective and injective

Examples: ...

Equivalence Relations

Equality-like & Sameness-like relations.

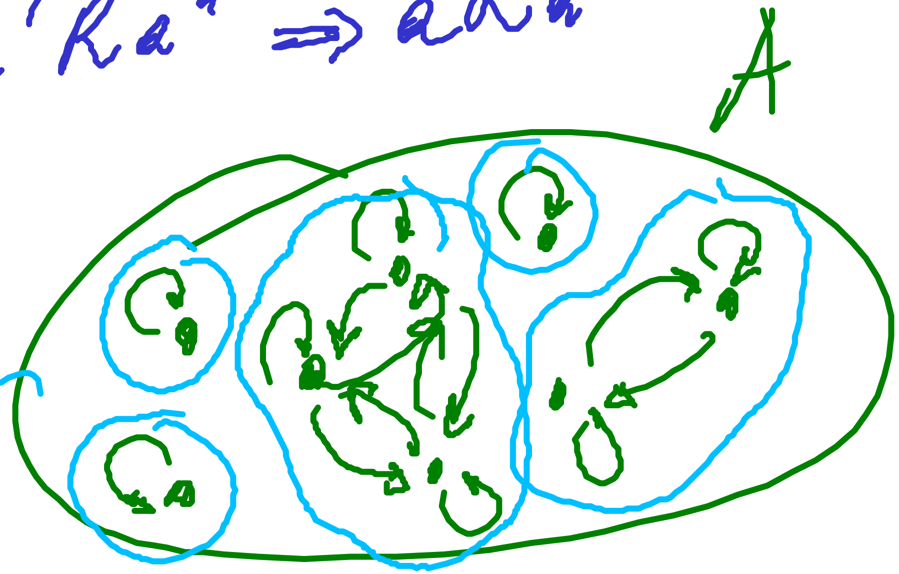
$R \subseteq A \times A$ is an equiv. relation whenever

(1) Reflexivity $\forall a \in A.\ aRa$

(2) Symmetric $\forall a, a' \in A.\ aRa' \Rightarrow a'Ra$

(3) Transitivity: $\forall a, a', a'' \in A$
$aRa' \wedge a'Ra'' \Rightarrow aRa''$

A Partition of A



Partitions

The set of all partitions of A

Part(A)

A **partition** P of A is a set of non-empty subsets of A, that is $P \subseteq \mathcal{P}(A)$, such that

referred to as blocks

(1) $\bigcup_{b \in P} b = A$

(2) $\forall b_1, b_2 \in P.\ b_1 \neq b_2 \Rightarrow b_1 \cap b_2 = \emptyset$

Idea

$P = \{b_1, b_2, \dots, b_n\}$

$b_i \cap b_j = \emptyset \ \forall i \neq j$

$\bigcup_i b_i = A$

$$\underline{\mathrm{EqRel}}(A) \underset{\varepsilon}{\overset{\pi}{\rightleftarrows}} \underline{\mathrm{Part}}(A)$$

set of equiv. relations on A

<u>Def</u> Let π be given by

$$\pi(R) \stackrel{\text{def}}{=} \text{- - - - . ...} \in \underline{\mathrm{Part}}(A)$$

for all $R \in \underline{\mathrm{EqRel}}(A)$.

<u>auxiliary def</u> For $a \in A$ define

the equiv. class of a: $[a] \stackrel{\text{def}}{=} \{ x \in A \mid aRx \}$

$$\stackrel{\text{def}}{=} \{ [a] \mid a \in A \}$$

Need to check that $\pi(R)$ is a partition

(1) $\bigcup_{a \in A} [a] = A$

PROPERTY $\forall a \in A \quad a \in [a]$

(2) $[a] \cap [b] \neq \emptyset \implies [a] = [b]$

$x \in [a] \cap [b] \implies x \in [a]$ and $x \in [b]$

$\implies x R a$ and $x R b$

$\implies a R b$ and $b R a$

EXERCISE

LEMMA:

aRb iff $[a] = [b]$

$\mathcal{E} : \underline{\mathrm{Part}}(A) \longrightarrow \underline{\mathrm{EqRel}}(A)$

for all partitions P of A define

$$\mathcal{E}(P) =^{\mathrm{def}} \left\{ (x,y) \;\middle|\; \begin{array}{l} \exists b \in P. \\ x \in b \text{ and } y \in b \end{array} \right\} \subseteq A \times A$$

is an equiv. rel.

(1) $\forall x.\ (x,x) \in \mathcal{E}(P)$ which holds because every element of A is in a block

because every element is in a unique block

(2) $(x,y) \in \mathcal{E}(P) \Longrightarrow (y,x) \in \mathcal{E}(P)$

(3) $(x,y) \in \mathcal{E}(P)$ and $(y,z) \in \mathcal{E}(P) \Longrightarrow (x,z) \in \mathcal{E}(P)$

$\Downarrow$ $\exists b.\ x \in b$ and $y \in b$

$\Downarrow$ $\exists b'.\ y \in b'$ and $z \in b'$