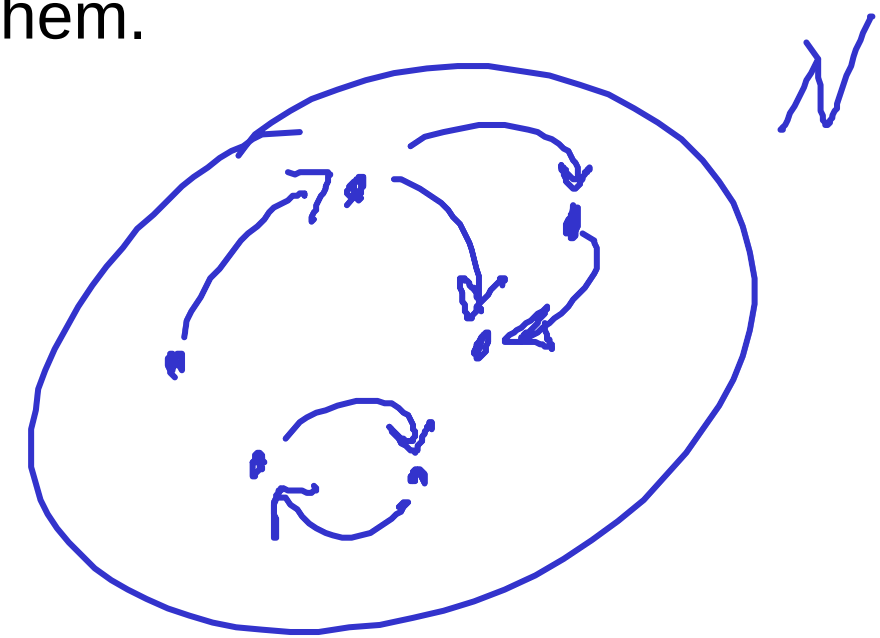
Directed Graphs

A *directed graph* is specified by a set of nodes and directed edges between them.



They can be therefore formalised as a set N together with a relation $E \subseteq N \times N$ on it.

For a directed graph $\mathsf{E} \subseteq \mathsf{N} \times \mathsf{N}$, what can we say about the directed graphs:

notation $E^2, E^3, \ldots$

$\boxed{?}$ $\mathsf{E} \circ \mathsf{E}$, $\mathsf{E} \circ \mathsf{E} \circ \mathsf{E}$, . . . ?

E^n?

$x E^2 y$ iff $\exists z.\ x E z \wedge z E y$

iff there is a path of length 2 in the directed graph from x to y.

$x E^3 y$ iff $\exists z.\ x E^2 z \wedge z E y$

iff x is connected to y by a path of length 3

By induction,

$x \, E^n \, y$ iff There is a path of length n from x to y in the directed graph E

$E^+ =_{df} \bigcup_{n \in \mathbb{N}} E^n$

$x \, E^+ \, y$ iff $\exists n \in \mathbb{N}. \ x \, E^n \, y$

iff There exists a path from x to y

Transitive closure of E.

$$E^{(1)} = E$$

$$E^{(n+1)} = E \vee \left(E \cdot E^{(n)}\right)$$

If the directed graph has a finite set of nodes say k then $E^{+} = E^{(k)}$

For a directed graph $E \subseteq N \times N$, what can we say about the directed graphs:

$\boxed{?}$ $E \circ E$, $E \circ E \circ E$, . . . ?

$\boxed{?}$ $E^{-1} \circ E$?

$E : N \nrightarrow N \qquad E^{-1} : N \nrightarrow N$

$x \to z \leftarrow y$

$x(E^{-1} \circ E)y \iff \exists z.\ z E^{-1} y \wedge x E z$

$\iff \exists x.\ y E z \wedge x E z$

$\mathrm{Sur}(A,B) \subseteq \mathrm{Fn}(A,B) \subseteq \mathrm{Pfn}(A,B) \subseteq \mathrm{Rel}(A,B)$

Partial Functions

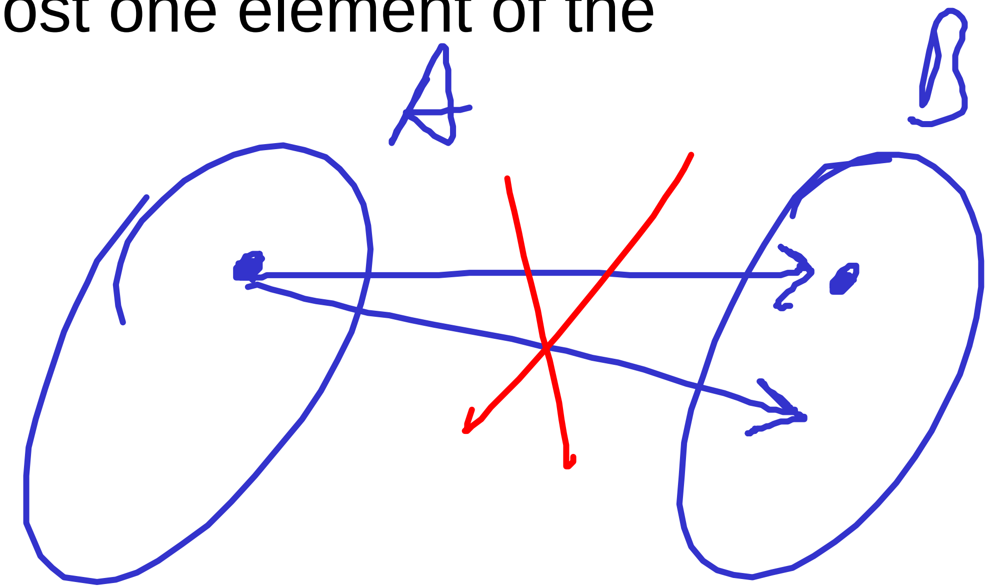
$\mathrm{Bij}(A,B) \subseteq \mathrm{Inj}(A,B)$ $\qquad$ $= \mathcal{P}(A \times B)$

- A *partial function* is a relation in which every element of the domain is related to at most one element of the codomain; that is, . . .

$f \in \mathcal{P}(A \times B)$ is a partial function when

$\forall a \in A.\ \forall b, b' \in B.$

$a \mathrel{f} b \wedge a \mathrel{f} b' \Rightarrow b = b'$



Examples: . . .

input/output behaviour.

The unique $b \in B$ to which $a \in A$ is related to, if at all, is denoted $f(a)$.

For a partial function f from A to B

$f(a)$
- could be undefined exactly when a is not related to anything in B
- or it denotes the unique element of B to which a is related by f.

A, B finite sets.

$$\#\mathcal{P}(A \times B) = 2^{\#(A\times B)} = 2^{\#A \cdot \#B}.$$

$$\#(A \rightharpoonup B) = ?$$

<u>Notation</u> $\underbrace{(A \rightharpoonup B),\ \mathrm{Pfn}(A,B)}$

└ denotes the set of partial functions from A to B

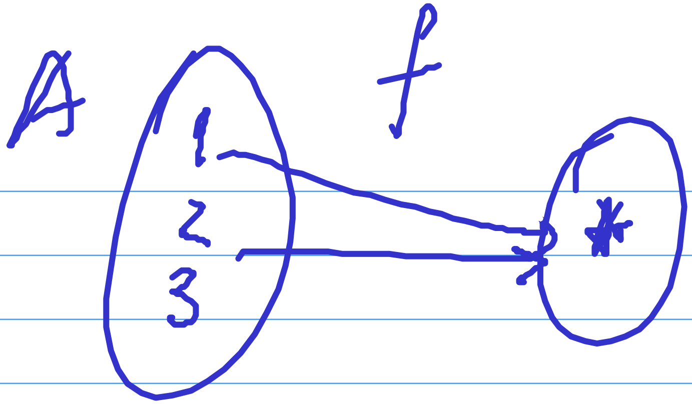
$\#(A \rightharpoonup \{*\}) = \#\mathcal{P}(A)$

$f : A \rightharpoonup \{*\} \rightsquigarrow \mathrm{dom}(f) \subseteq A$

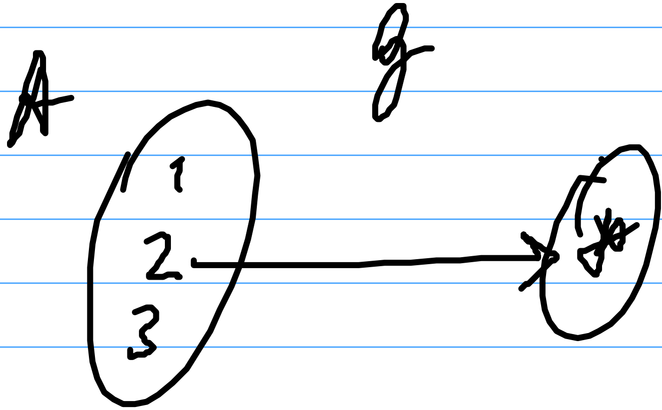
Def: $\underline{\mathrm{dom}}(f) \subseteq A$, the domain of definition of f,

$\|\mathrm{def}$

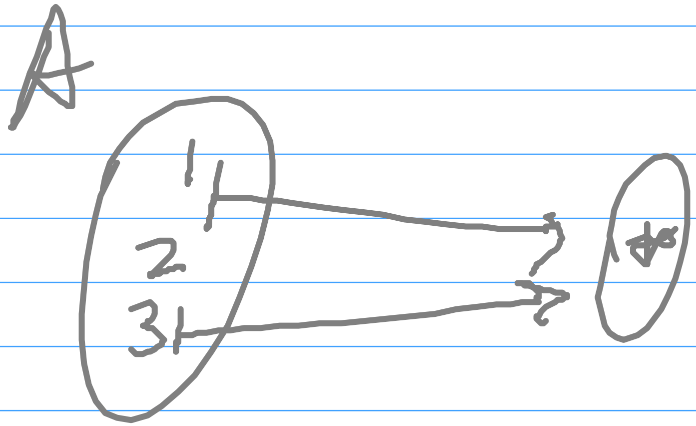
$\{a \in A \mid f(a) \text{ is defined}\}$



$\mathrm{dom}\, f = \{1,2\}$



$\mathrm{dom}\, g = \{2\}$



$\{1,3\}$

$$\#(A \rightharpoonup \{*\}) = \#\mathcal{P}(A) = 2^{\#A}$$

$$\#\mathcal{P}(A) = 2^{\#A}$$

$$\|$$

$$\sum_k \# \underbrace{\mathcal{P}^{(k)}(A)}$$

the subsets of A of size exactly k

Let $\#A = n$

$$\|$$

$$\sum_k \binom{n}{k}$$

$C^n_k \quad {}_kC^n$

|| binomial Theorem

$$2^n$$

$$(x+y)^n = \sum_k \binom{n}{k} x^k y^{n-k}$$

$\boxed{?}$ Is the relational composition of two partial functions a partial function?

$f : A \to B \qquad g : B \to C$

$g \circ f \in \mathcal{P}(A \times C)$

$\boxed{?}$ Is it a partial function?

$\boxed{!}$ yes!

$(g \circ f)(x)$ — undef. if $x \notin \underline{\mathrm{dom}}(f)$

if $x \in \underline{\mathrm{dom}}(f)$:

- undef. if $f(x) \notin \underline{\mathrm{dom}}(g)$
- $g(f(x))$ if $f(x) \in \underline{\mathrm{dom}}(g)$

(Total) Functions

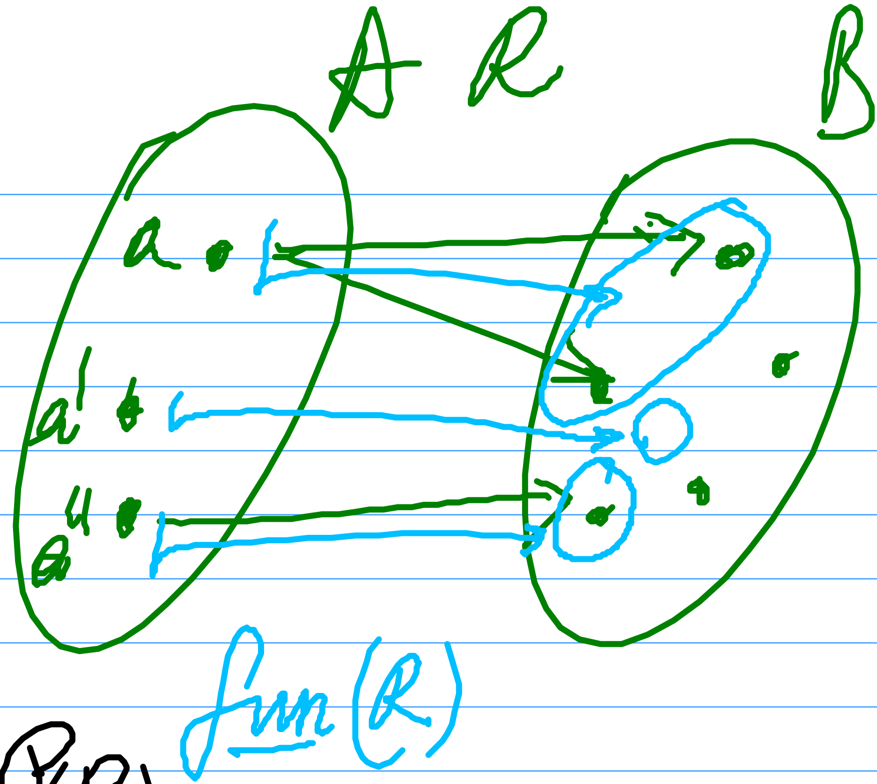
- A *function* is a partial function defined on every element of the domain; that is, . . .

Def A function f from A to B (notation $f : A \to B$) is a partial function $f : A \to B$ such that $dom(f) = A$

Examples: . . . Every relation R from A to B induces a function from A to $\mathcal{P}(B)$

$R \subseteq A \times B$

$\underline{\text{fun}}(R) : A \to \mathcal{P}(B)$



$\forall a \in A.$

$$\underline{\text{fun}}(R)(a) \overset{\text{def}}{=} \{ b \in B \mid aRb \} \in \mathcal{P}(B)$$

Given $f : A \to \mathcal{P}(B)$

$\underline{\text{rel}}(f) \subseteq A \times B$

def $\forall a \in A, b \in B, \ (a,b) \in \underline{\text{rel}}(f) \text{ iff}_{\text{def}} \ b \in f(a)$

$\underline{\text{rel}}(\underline{\text{fun}}(R)) = ?$

$\underline{\text{fun}}(\underline{\text{rel}}(f)) = ?$