Notations: $R \in \underline{\underline{Rel}}(A,B)$, $R : A \nrightarrow B$

Relations

- A *relation* R from a set A to a set B is a subset of the product set $A \times B$; that is,

$$R \subseteq A \times B$$

or equivalently

$$R \in \mathcal{P}(A \times B)$$

NB: Relations come with a *domain* and a *codomain*.

Examples: . . .

Examples: (1) $\emptyset : A \nrightarrow B$ **empty relation**

(a, b) is never in the empty relation

(2) $A \times B : A \nrightarrow B$ **full relation**

(a, b) is always in the full relation.

(3) $S : \mathbb{N} \nrightarrow \mathbb{N}$

$n \, S \, m$ iff $m = n^2$

(4) Computation defines a relation

(5) Network define a relation.

$\boxed{?}$ Given a relation from A to B, is there a natural way in which to induce a relation from B to A?

$$A \xrightarrow{R} B \quad \overset{?}{\rightsquigarrow} \quad B \xrightarrow{R^{-1}} A$$

$\underline{\text{def}}$ $b \in B,\ a \in A$

$b \, R^{-1} \, a$ iff$_{\text{def}}$ $a \, R \, b$

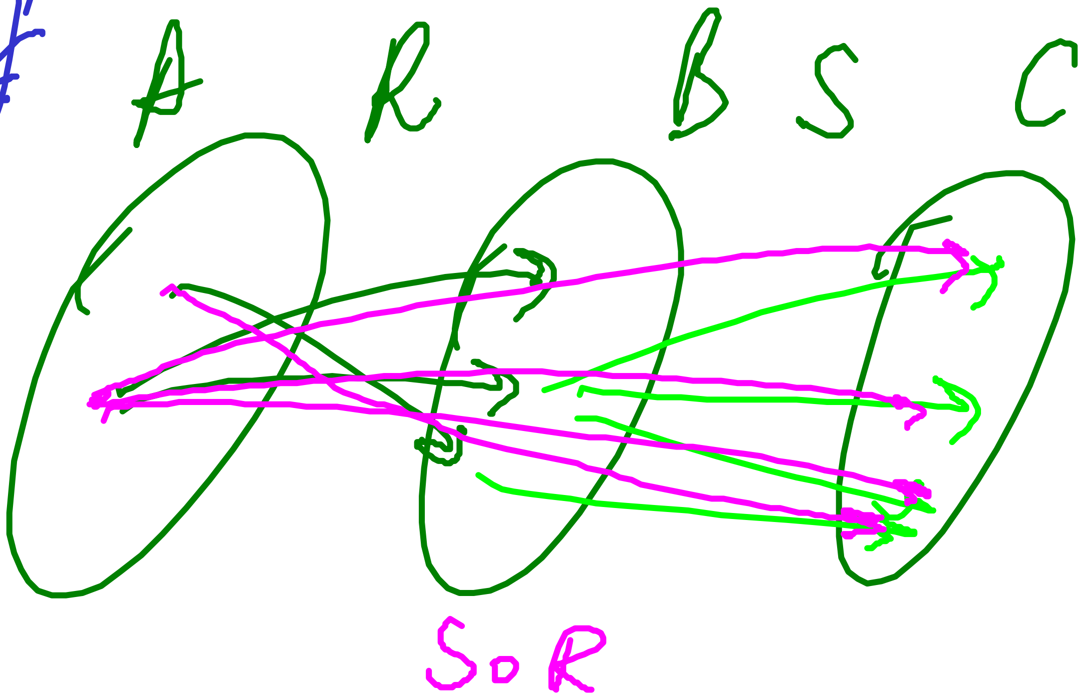
Composition

notation $R \,;\, S$

Given $R \subseteq A \times B$ and $S \subseteq B \times C$ we wish to define $S \circ R \subseteq A \times C$.

$A \overset{R}{\nrightarrow} B \qquad B \overset{S}{\nrightarrow} C \quad \rightsquigarrow \quad A \overset{S \circ R}{\nrightarrow} C$

Def



$a \in A,\ c \in C$

$a \, (S \circ R) \, c$

iff. $\exists b \in B.$

$a R b$

$\wedge$

$b S c$

? What can we say about the two different ways in which one can define the composition of three relations?

$$\begin{tikzcd} A \arrow[r, "R"] \arrow[rr, bend right, "S \circ R"'] & B \arrow[r, "S"] \arrow[rr, bend left, "T \circ S"] & C \arrow[r, "T"] & D \end{tikzcd}$$

$$(T \circ S) \circ R \overset{?}{=} T \circ (S \circ R)$$

? Does composition has a neutral element?

$I_X : X \nrightarrow X \quad \forall X$

For $R : A \nrightarrow B$:

$$R \circ I_A = R = I_B \circ R$$

$x \, I_X \, y \text{ iff}_{def} \, x = y$

Identity Relation

ASSOCIATIVITY OF COMPOSITION

$$(T \circ S) \circ R = T \circ (S \circ R) : A \leftrightarrow D$$

Show $\forall a \in A, d \in D$.

$$a\,[(T \circ S) \circ R]\,d \text{ iff } a\,[T \circ (S \circ R)]\,d$$

$a\,[(T \circ S) \circ R]\,d$ iff $\exists b.\ aRb \wedge b\,(T \circ S)\,d$

iff $\exists b.\ aRb \wedge (\exists c.\ bSc \wedge cTd)$

Check this out

iff $\exists c.\ (\exists b\ aRb \wedge bSc) \wedge cTd$

iff $\exists c.\ a\,(S \circ R)\,c \wedge cTd$

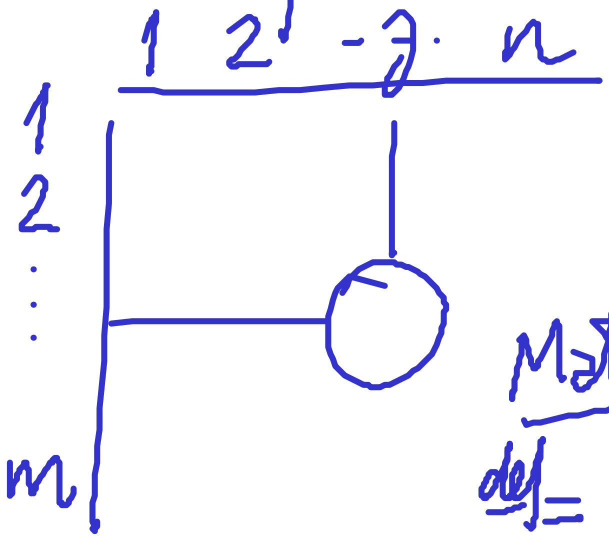
iff $a\,(T \circ (S \circ R))\,d$ $\boxed{X}$

$\underline{k} = \{1, 2, \ldots, k\}$

Relations as Matrices

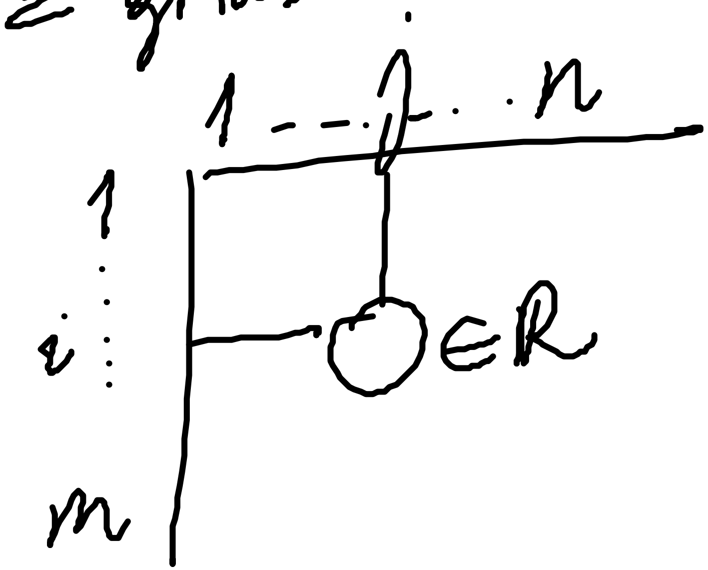
$R : \underline{m} \nrightarrow \underline{n}$

in matrix form:



$\underline{Mat}(R)_{i,j} \stackrel{def}{=}$ true iff iRj

└ 2-dimensional vectors ≡ grids.



$M \in \mathbb{R}^{m \times n}$

$L \in \mathbb{R}^{n \times l}$

$\rightsquigarrow$ $(M \cdot L) \in \mathbb{R}^{m \times l}$

$$(M \cdot L)_{i,j} = \sum_k L_{k,j} * M_{i,k}$$

Given $R : \underline{m} \leftrightarrow \underline{n} \rightsquigarrow \underline{Mat}(R) \in \underline{Bool}^{m \times n}$

$\underline{Rel}(M) : m \leftrightarrow n \leftarrow\!\!\!\rightsquigarrow M \in \underline{\underline{Bool}}^{m \times n}$

$(i,j) \in \underline{Rel}(M)$
iff $\underline{def}$ $M_{i,j} = true$

$\underline{Rel}(\underline{Mat}(R)) = R$

$\underline{Mat}(\underline{Rel}(M)) = M$

a bijective or 1-1 correspondence

inverse $\longleftrightarrow$ transposition composition $\longleftrightarrow$ multiplication

for real valued matrices

$R : \underline{m} \to \underline{n}$, $L : \underline{n} \to \underline{\ell}$

$$(M \cdot L)_{i,j} = \sum_k L_{kj} \times M_{ik}$$

$(L \circ R) : \underline{m} \to \underline{\ell}$

$M \in \mathbb{R}^{m \times n}$, $L \in \mathbb{R}^{n \times \ell}$

$i \, (L \circ R) \, j$

iff $\exists k \in \underline{n}.$

$(k L j) \wedge (i R k)$

$(L \cdot M) \in \mathbb{R}^{m \times \ell}$

iff $\bigvee_{k \in \underline{n}} (k L j) \wedge (i R k)$

$M \in Bool^{m \times n}$, $L \in Bool^{n \times \ell}$

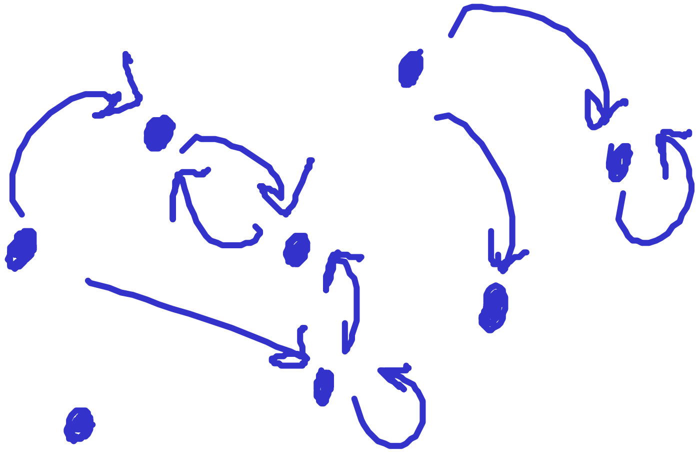
$(L \cdot M) \in Bool^{m \times \ell}$

$$(L \cdot M)_{i,j} = \bigvee_k L_{kj} \wedge M_{ik}$$

Directed Graphs

or vertices

A *directed graph* is specified by a set of nodes and directed edges between them.



is a relation from a set to itself

That is, $E : N \to N$

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