- Comprehension:

Bounded Comprehension

$$\{x \mid P(x)\} \quad , \quad \{x \in X \mid P(x)\}$$

} for X a set.

cannot be arbitrary so as to define sets.

(cf. DIAGONALISATION)

Take $P(x) =_{def} (x \notin x)$

Suppose that $R =_{def} \{x \mid x \notin x\}$ is a set.

Is $R \in R$? Suppose so. Then $R \notin R$ ↯

Is $R \notin R$? Suppose so. Then it is not the case $R \notin R$

That is, $R \in R$ ↯

- Membership, inclusion, equality:

$$x \in X \quad , \quad X \subseteq Y, \quad , \quad X = Y$$

$a \in \{x \mid P(x)\}$ iff $P(a)$

$X \subseteq Y$ & $Y \subseteq X$

$\forall x \in X.\ x \in Y$

- Powerset:

$\mathcal{P}(U)$

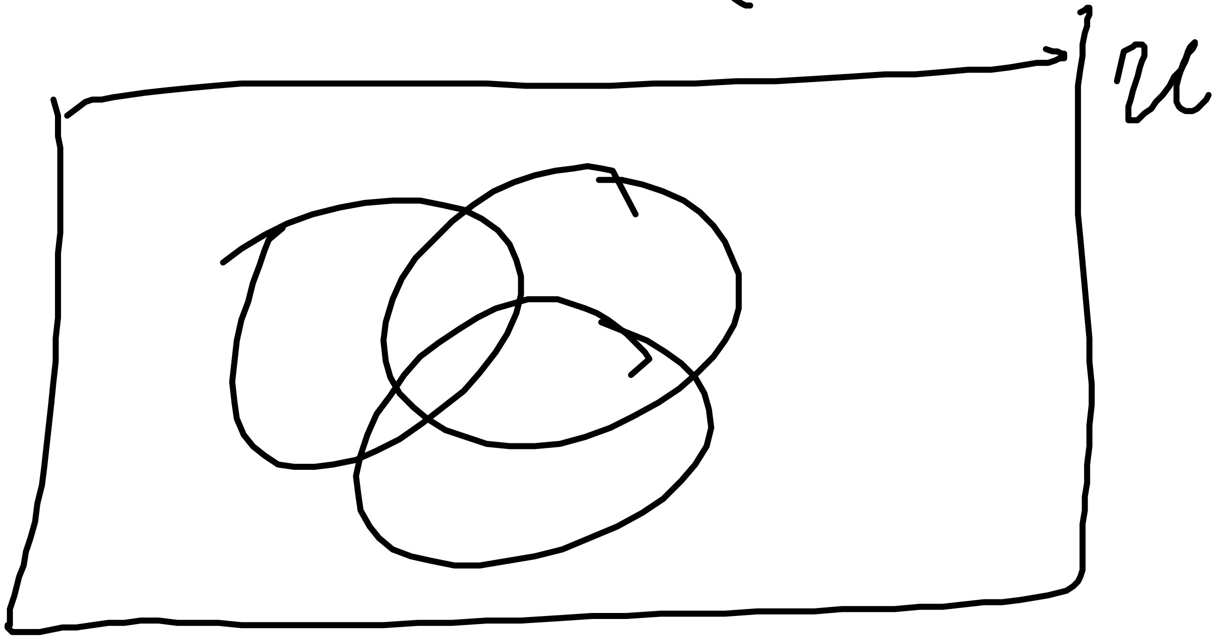
If U is a set then $\mathcal{P}(U)$ is a set

$$=^{def} \{ S \mid S \subseteq U \}$$

- If U is finite then $\mathcal{P}(U)$ is finite. ($\#\mathcal{P}(U) = 2^{\# U}$)

cardinality or size

- Venn diagrams.

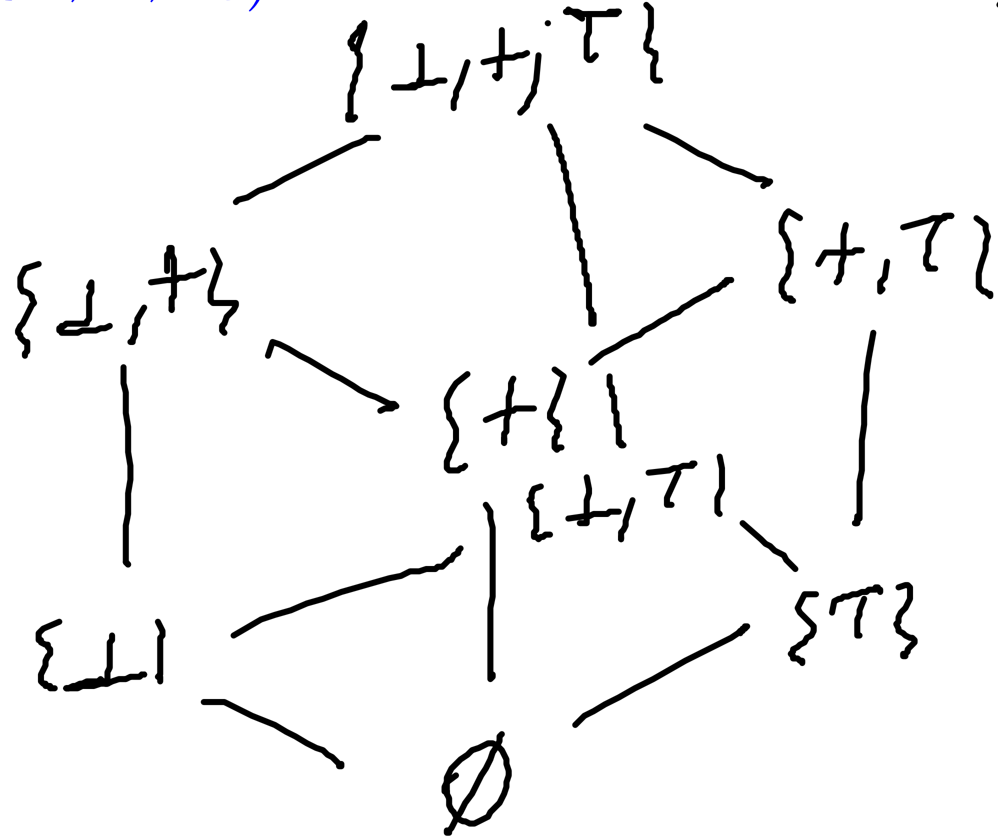


- $\#\mathcal{P}(U) > 0$
- $\emptyset, U \in \mathcal{P}(U)$

Example: $\mathcal{P}(\{\perp, +, \top\})$

~ cardinality $2^3 = 8$

Hasse diagram



What about

$\mathcal{P}(\mathcal{P}(\mathcal{P}(\{\perp, +, \top\})))$?

constants or nullary operations

The Boolean algebra of sets

$$(\quad \mathcal{P}(\mathcal{U}), \quad \emptyset, \quad \mathcal{U}, \quad \cup, \quad \cap, \quad (\cdot)^c \quad)$$

binary operation

unary operation

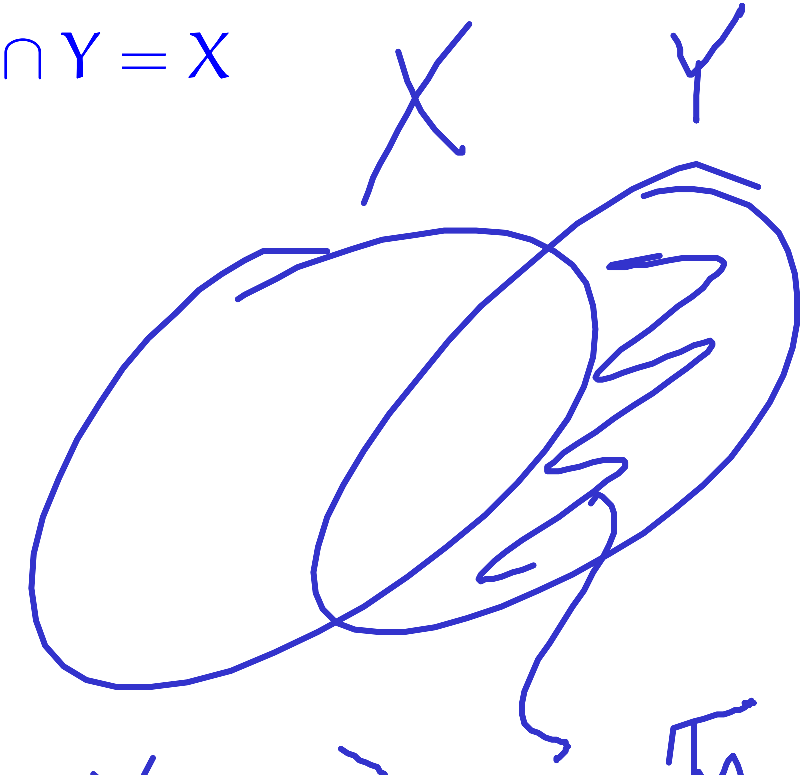
$$X^c = \{ x \in \mathcal{U} \mid x \notin X \}$$

The Boolean algebra of sets

$$\left(\quad \mathcal{P}(U) \; , \quad \emptyset \; , \quad U \; , \quad \cup \; , \quad \cap \; , \quad (\cdot)^c \quad \right)$$

NB: For all $X, Y \in \mathcal{P}(U)$,

$$X \cup Y = Y \;\text{ iff }\; X \subseteq Y \;\text{ iff }\; X \cap Y = X$$



$X \cap Y = X \Rightarrow$ empty

- The union operation $\cup$ and the intersection operation $\cap$ are associative, commutative, and idempotent.

$$(A \cup B) \cup C = A \cup (B \cup C)$$
$$(A \cap B) \cap C = A \cap (B \cap C)$$

$\Rightarrow$ the expression $A_1 \cap A_2 \cap A_3 \cap A_4$ is not ambiguous

$$A \cup B = B \cup A, \quad A \cup A = A$$
$$A \cap B = B \cap A \quad A \cap A = A$$

- The empty set $\emptyset$ is a neutral element for $\cup$ and the universal set $\mathcal{U}$ is a neutral element for $\cap$.

$\emptyset \cup X = X$ $\qquad$ $\mathcal{U} \cap X = X$

- With respect to each other, the union operation $\cup$ and the intersection operation $\cap$ are absorptive and distributive.

$$X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$$

$$X = X \cup (X \cap A)$$

$$X = X \cap (X \cup A)$$

- The complement operation $(\cdot)^c$ satisfies complementation laws.

$$X^c \cup X = \mathcal{U}$$

$$X^c \cap X = \emptyset$$

NB: For all $X, Y \in \mathcal{P}(\mathcal{U})$,

$$X^c = Y \text{ iff } (X \cup Y = \mathcal{U} \text{ and } X \cap Y = \emptyset)$$

($\Rightarrow$) $X \cup X^c = \mathcal{U}$ and $X \cap X^c = \emptyset$ by definition

($\Leftarrow$) If you show that Y has the property of complementation w.r.t. to X then actually Y is X^c.

De Morgan's Laws

To show

$$(X \cup Y)^c = X^c \cap Y^c$$

it is enough to show

$$(X \cup Y) \cup (X^c \cap Y^c) = \mathcal{U}$$

and

$$(X \cup Y) \cap (X^c \cap Y^c) = \emptyset$$

EXERCISE

$A^c = B$

iff

$A \cup B = \mathcal{U}$

and

$A \cap B = \emptyset$.

Sets and Logic

$\mathcal{P}(U)$	$\{0,1\}$
$\emptyset$	0
U	1
$\cup$	$\vee$
$\cap$	$\wedge$
$(\cdot)^c$	$\neg(\cdot)$

$\mathcal{P}(U) = \{\emptyset, \{*\}\}$

U a singleton $\{*\}$.

Chapter 3

Reading list:

3.1 Ordered pairs and products

3.2 Relations and functions

3.3 Relations as structure

3.3.1 Directed graphs

3.3.2 Equivalence relations

3.4 Size of sets

Suggested exercises: 3.1, 3.4, 3.5, 3.6, 3.10, 3.12, 3.17, 3.18, 3.21, 3.33, 3.36.

Product of Sets

Ordered pairs:

The *ordered pairing* of a and b is denoted (a, b).

NB: $(a, b) = (x, y)$ iff $a = x$ and $b = y$

Product of Sets

Ordered pairs:

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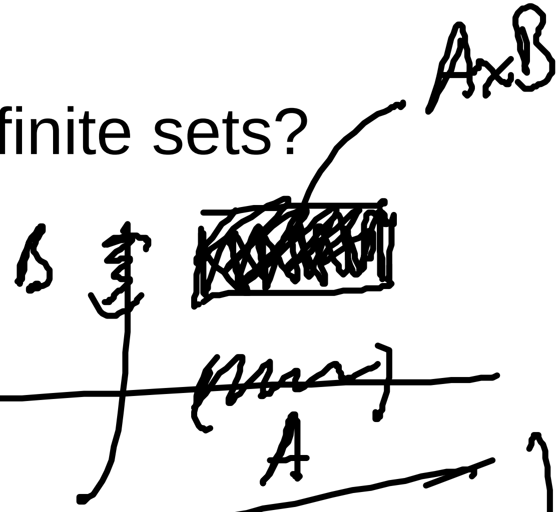
Product construction:

$$A \times B =_{\text{def}} \{ (a, b) \mid a \in A \text{ and } b \in B \}$$

$\#(A \times B) = \#A \cdot \#B$. finite case

$\boxed{\textbf{?}}$ What is the cardinality of the product of two finite sets?

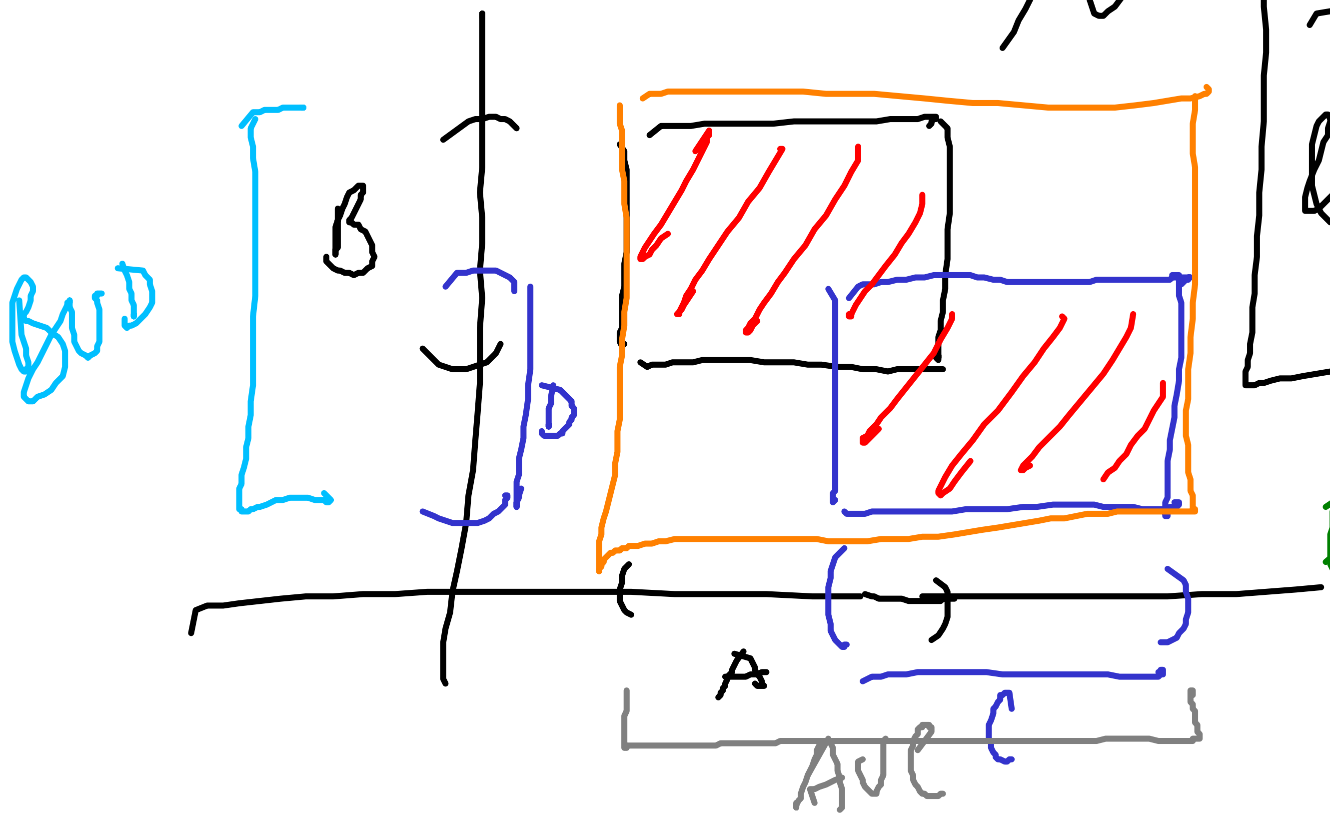
? What is the cardinality of the product of two finite sets?



? $(A \times B) \cup (C \times D) \overset{?}{=} (A \cup C) \times (B \cup D)$?

Conjecture

EXERCISE

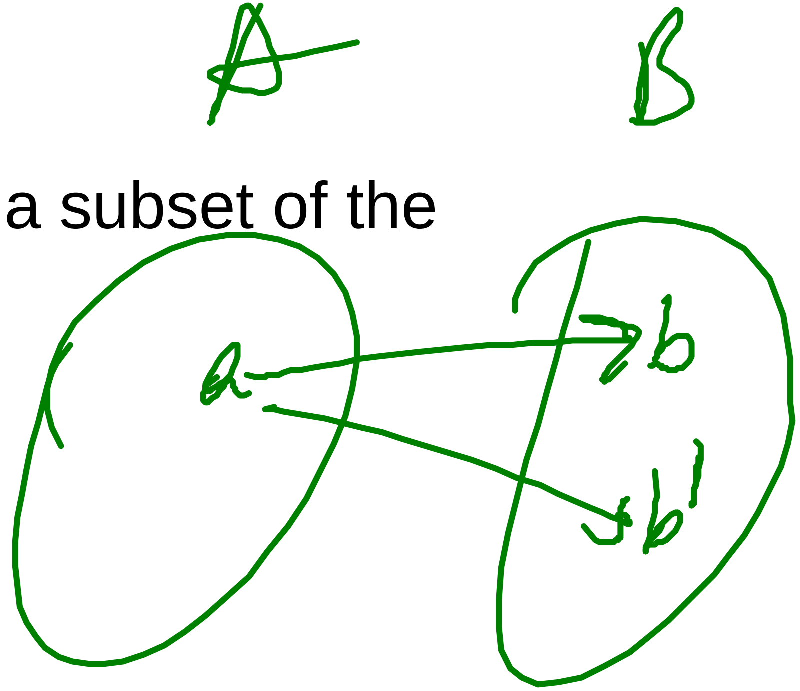


notation aRb

$(a,b) \in R$

$(a,b') \in R$

Relations



- A *relation* R from a set A to a set B is a subset of the product set $A \times B$; that is,

notation $R : A \nrightarrow B$

$$R \subseteq A \times B$$

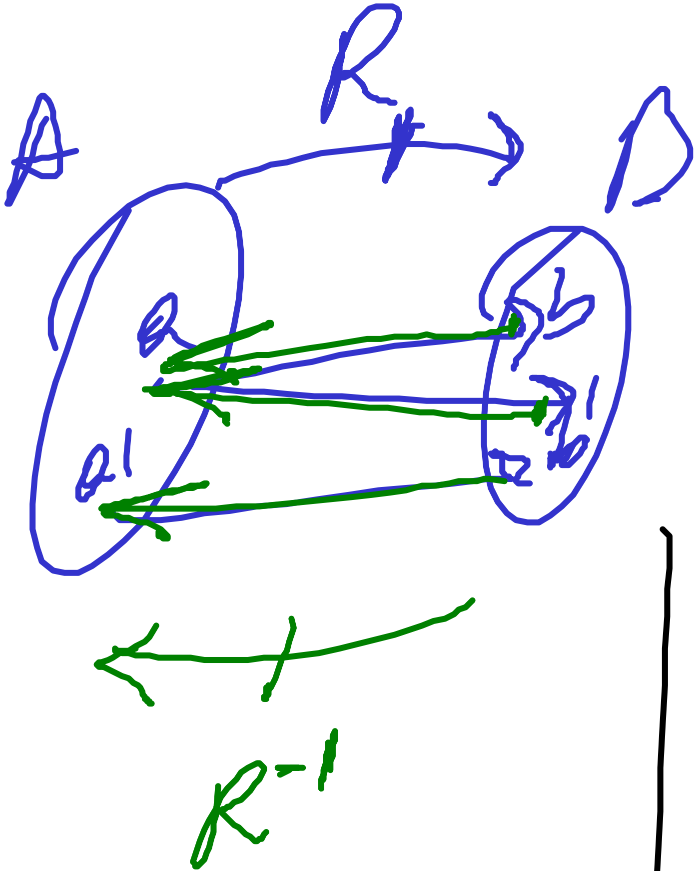
or equivalently

$$R \in \mathcal{P}(A \times B)$$

NB: Relations come with a *domain* or source and a *codomain* or target.

Examples: ...

? Given a relation from A to B, is there a natural way in which to induce a relation from B to A?



Inverse relation

notation $\quad B \xrightarrow{R^{-1}} A$

Def $R^{-1} : B \nrightarrow A$

$b \, R^{-1} \, a \iff_{\text{def}} a \, R \, b$