Topic 4

Scott Induction

Scott's Fixed Point Induction Principle

Let $f : D \to D$ be a continuous function on a domain D.

For any admissible subset $S \subseteq D$, to prove that the least fixed point of f is in S, *i.e.* that

$$\mathit{fix}(f) \in S \ ,$$

it suffices to prove

$$\forall d \in D \ (d \in S \ \Rightarrow \ f(d) \in S) \ .$$

Chain-closed and admissible subsets

Let D be a cpo. A subset $S \subseteq D$ is called chain-closed iff for all chains $d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \ldots$ in D

$$(\forall n \geq 0 \,.\, d_n \in S) \;\Rightarrow\; \left(\bigsqcup_{n \geq 0} d_n \right) \in S$$

If D is a domain, $S \subseteq D$ is called admissible iff it is a chain-closed subset of D and $\perp \in S$.

$\forall$ admissible S.

$(\forall d \,.\, d \in S \Rightarrow f\,d \in S) \Rightarrow fix(f) \in S$

Assume $\quad \forall d.\ d \in S \Rightarrow f\,d \in S$

Show $\quad \underline{fix}(f) \in S$

$$\bot \in S \Rightarrow f(\bot) \in S \Rightarrow f^2(\bot) \in S \Rightarrow \cdots f^n(\bot) \in S$$

$$\bot \sqsubseteq f(\bot) \sqsubseteq \cdots \sqsubseteq f^n(\bot) \sqsubseteq \cdots \in S$$

$$\Rightarrow \quad fix(f) = \bigsqcup_n f^n(\bot) \in S.$$

$$\frac{\forall d.\quad d \in S \Rightarrow f(d) \in S}{\underline{\mathrm{fix}}(f) \in S} \quad (S \text{ admissible})$$

Chain-closed and admissible subsets

Let D be a cpo. A subset $S \subseteq D$ is called chain-closed iff for all chains $d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \ldots$ in D

$$(\forall n \geq 0 \, . \, d_n \in S) \;\Rightarrow\; \Big(\bigsqcup_{n\geq 0} d_n\Big) \in S$$

If D is a domain, $S \subseteq D$ is called admissible iff it is a chain-closed subset of D and $\bot \in S$.

A property $\Phi(d)$ of elements $d \in D$ is called *chain-closed* (resp. *admissible*) iff $\{d \in D \mid \Phi(d)\}$ is a *chain-closed* (resp. *admissible*) subset of D.

Building chain-closed subsets (I)

Let D, E be cpos.

$$\frac{\forall x.\ x \in \downarrow d \Rightarrow f(x) \in \downarrow(d)}{\mathit{fix}(f) \in \downarrow(d)}$$

Basic relations:

- For every $d \in D$, the subset

$$\downarrow(d) \stackrel{\text{def}}{=} \{\, x \in D \mid x \sqsubseteq d \,\}$$

of D is chain-closed.

$d_0 \sqsubseteq d_1 \sqsubseteq \cdots \sqsubseteq d_n \sqsubseteq \cdots \quad \in \downarrow(d)$

(lub2)

$\Longrightarrow \bigsqcup_n d_n \in \downarrow(d)$

Building chain-closed subsets (I)

Let D, E be cpos.

$x_0 \sqsubseteq x_1 \sqsubseteq \cdots \sqsubseteq x_n \sqsubseteq \cdots$

$y_0 \sqsubseteq y_1 \sqsubseteq \cdots \sqsubseteq y_n \sqsubseteq \cdots$

Basic relations:

- For every $d \in D$, the subset

$$\downarrow(d) \stackrel{\text{def}}{=} \{\, x \in D \mid x \sqsubseteq d \,\}$$

  of D is chain-closed.

$(x_0, y_0) \sqsubseteq \cdots \sqsubseteq (x_n, y_n) \sqsubseteq \cdots$

s.t. $x_0 \sqsubseteq y_0,\ x_1 \sqsubseteq y_1, \cdots\ x_n \sqsubseteq y_n \cdots$

- The subsets

$$\{(x, y) \in D \times D \mid x \sqsubseteq y\}$$

  and

$$\{(x, y) \in D \times D \mid x = y\}$$

  of $D \times D$ are chain-closed.

$\Rightarrow \bigsqcup_n (x_n, y_n) = (\bigsqcup_n x_n, \bigsqcup_n y_n)$

is such that

$\bigsqcup_n x_n \sqsubseteq \bigsqcup_n y_n$

Example (I): Least pre-fixed point property

Let D be a domain and let $f : D \to D$ be a continuous function.

$$\forall d \in D.\, f(d) \sqsubseteq d \implies \mathit{fix}(f) \sqsubseteq d$$

Look at the admissible set $\downarrow(d)$

Example (I): Least pre-fixed point property

Let D be a domain and let $f : D \to D$ be a continuous function.

$$\forall d \in D.\, f(d) \sqsubseteq d \implies \mathit{fix}(f) \sqsubseteq d$$

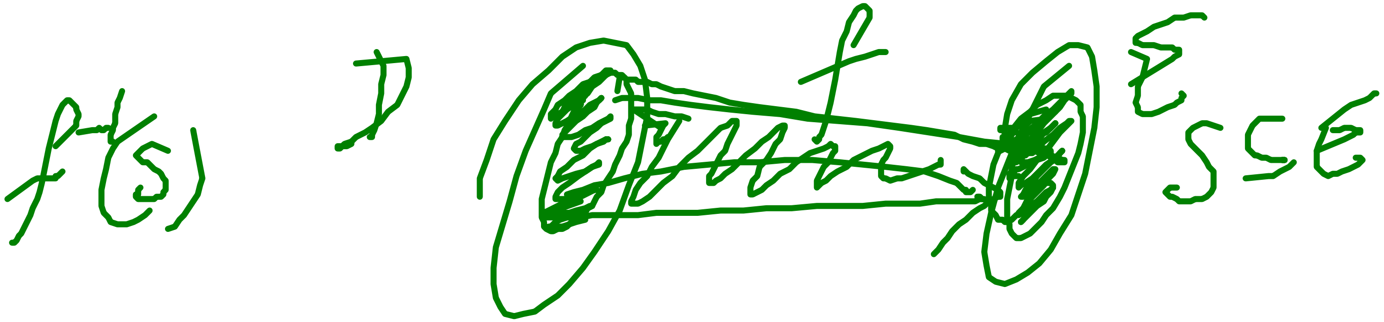
Proof by Scott induction.

Let $d \in D$ be a pre-fixed point of f. Then,

$$\begin{aligned} x \in {\downarrow}(d) \quad & \implies \quad x \sqsubseteq d \\ & \implies \quad f(x) \sqsubseteq f(d) \\ & \implies \quad f(x) \sqsubseteq d \\ & \implies \quad f(x) \in {\downarrow}(d) \end{aligned}$$

Hence,

$$\mathit{fix}(f) \in {\downarrow}(d) \ .$$



Building chain-closed subsets (II)

Inverse image:

Let $f : D \to E$ be a continuous function.

If S is a chain-closed subset of E then the inverse image

$$f^{-1}S \;=\; \{x \in D \mid f(x) \in S\}$$

is an chain-closed subset of D.

? $d_0 \sqsubseteq d_1 \sqsubseteq \cdots \sqsubseteq d_n \sqsubseteq \cdots$ in $f^{-1}(S)$

$\Rightarrow \bigsqcup_n d_n \in f^{-1}(S) \iff f(\bigsqcup_n d_n) \in S$

$= \bigsqcup_n f(d_n)$

Example (II)

Let D be a domain and let $f, g : D \to D$ be continuous functions such that $f \circ g \sqsubseteq g \circ f$. Then,

$$f(\bot) \sqsubseteq g(\bot) \implies \mathit{fix}(f) \sqsubseteq \mathit{fix}(g)\ .$$

Assume $f(gx) \sqsubseteq g(fx)\ \forall x.$ $\quad f(\bot) \sqsubseteq g(\bot)$

Know $\mathit{fix}(f) = \bigsqcup_n f^n(\bot)$ $\quad \mathit{fix}(g) = \bigsqcup_n g^n(\bot)$

Show

$$\frac{f^n \bot \overset{?}{\sqsubseteq} g^n \bot}{\bigsqcup_n f^n \bot \underset{?}{\sqsubseteq} \bigsqcup_n g^n(\bot)}$$

Show $f^n(\bot) \sqsubseteq g^n(\bot)$

$n=0$ $\bot \sqsubseteq \bot$ ✓

$n=1$ $f\bot \sqsubseteq g\bot$ by assumption ✓

$ff\bot \sqsubseteq fg\bot \sqsubseteq gf\bot \sqsubseteq gg\bot$

by induction on n

Example (II)

Let D be a domain and let $f, g : D \to D$ be continuous functions such that $f \circ g \sqsubseteq g \circ f$. Then,

$$f(\bot) \sqsubseteq g(\bot) \implies \mathit{fix}(f) \sqsubseteq \mathit{fix}(g) \ .$$

Proof by Scott induction.

Consider the admissible property $\Phi(x) \equiv \big(f(x) \sqsubseteq g(x)\big)$ of D.

$\hookrightarrow$ $\Phi(\bot)$ holds by assumption

Since

$$f(x) \sqsubseteq g(x) \Rightarrow g(f(x)) \sqsubseteq g(g(x)) \Rightarrow f(g(x)) \sqsubseteq g(g(x))$$

we have that $\Phi(x) \Rightarrow \Phi(gx)$

$$f(\mathit{fix}(g)) \sqsubseteq g(\mathit{fix}(g)) \ . \implies \mathit{fix}(f) \sqsubseteq \mathit{fix}(g)$$

$\Phi(\mathit{fix}(g))$ Ex.

Building chain-closed subsets (III)

Logical operations:

- If $S, T \subseteq D$ are chain-closed subsets of D then

  $$S \cup T \qquad \text{and} \qquad S \cap T$$

  are chain-closed subsets of D.

- If $\{ S_i \}_{i \in I}$ is a family of chain-closed subsets of D indexed by a set I, then $\bigcap_{i \in I} S_i$ is a chain-closed subset of D.

- If a property $P(x, y)$ determines a chain-closed subset of $D \times E$, then the property $\forall x \in D.\ P(x, y)$ determines a chain-closed subset of E.

Example (III): Partial correctness

Let $\mathcal{F} : State \rightharpoonup State$ be the denotation of

$$\mathbf{while}\ X > 0\ \mathbf{do}\ (Y := X * Y ; X := X - 1)\ .$$

The output value is defined.

For all $x, y \geq 0$,

$$\mathcal{F}[X \mapsto x, Y \mapsto y] \downarrow$$

$$\implies \mathcal{F}[X \mapsto x, Y \mapsto y] = [X \mapsto 0, Y \mapsto !x \cdot y].$$

Recall that

$$\mathcal{F} = \mathit{fix}(f)$$

where $f : (State \rightharpoonup State) \to (State \rightharpoonup State)$ is given by

$$f(w) = \lambda(x, y) \in State. \begin{cases} (x, y) & \text{if } x \leq 0 \\ w(x - 1, x \cdot y) & \text{if } x > 0 \end{cases}$$

Wanted to show $\mathrm{fix}(f) \in S$

Proof by Scott induction.

We consider the admissible subset of $(State \rightharpoonup State)$ given by

$$S = \left\{ w \;\middle|\; \begin{array}{l} \forall x, y \geq 0. \\ \quad w[X \mapsto x, Y \mapsto y]\downarrow \\ \qquad \Rightarrow w[X \mapsto x, Y \mapsto y] = [X \mapsto 0, Y \mapsto !x \cdot y] \end{array} \right\}$$

and show that

$$w \in S \implies f(w) \in S \; .$$

$\perp_{(State \rightharpoonup State)} \in S$

Topic 5

PCF

PCF syntax

Types

$$\tau ::= nat \mid bool \mid \tau \to \tau$$

$f : \tau_1 * \tau_2 \to \tau$

$f' : \tau_1 \to \tau_2 \to \tau$

$f'' : \tau_2 \to \tau_1 \to \tau$

PCF syntax

Types

$$\tau ::= nat \mid bool \mid \tau \to \tau$$

Expressions

$$M \quad ::= \quad \mathbf{0} \mid \mathbf{succ}(M) \mid \mathbf{pred}(M)$$

PCF syntax

Types

$$\tau ::= nat \mid bool \mid \tau \to \tau$$

Expressions

$$\begin{array}{rcl} M & ::= & \mathbf{0} \mid \mathbf{succ}(M) \mid \mathbf{pred}(M) \\ & \mid & \mathbf{true} \mid \mathbf{false} \mid \mathbf{zero}(M) \end{array}$$

Testing whether the nat M is zero or not.

PCF syntax

Types

$$\tau ::= nat \mid bool \mid \tau \to \tau$$

Expressions

$$\begin{array}{rcl} M & ::= & \mathbf{0} \mid \mathbf{succ}(M) \mid \mathbf{pred}(M) \\ & \mid & \mathbf{true} \mid \mathbf{false} \mid \mathbf{zero}(M) \\ & \mid & x \mid \mathbf{if}\ M\ \mathbf{then}\ M\ \mathbf{else}\ M \end{array}$$

PCF syntax

Types

$$\tau ::= nat \mid bool \mid \tau \to \tau$$

Expressions

$$\begin{array}{rcl} M & ::= & \mathbf{0} \mid \mathbf{succ}(M) \mid \mathbf{pred}(M) \\ & \mid & \mathbf{true} \mid \mathbf{false} \mid \mathbf{zero}(M) \\ & \mid & x \mid \mathbf{if}\ M\ \mathbf{then}\ M\ \mathbf{else}\ M \\ & \mid & \mathbf{fn}\, x : \tau \,.\, M \mid M\, M \mid \mathbf{fix}(M) \end{array}$$

functions

where $x \in \mathbb{V}$, an infinite set of variables.

λ-expressions

application

PCF syntax

Types

$$\tau ::= nat \mid bool \mid \tau \to \tau$$

Expressions

$$\begin{array}{rcl} M & ::= & \mathbf{0} \mid \mathbf{succ}(M) \mid \mathbf{pred}(M) \\ & \mid & \mathbf{true} \mid \mathbf{false} \mid \mathbf{zero}(M) \\ & \mid & x \mid \mathbf{if}\ M\ \mathbf{then}\ M\ \mathbf{else}\ M \\ & \mid & \mathbf{fn}\, x : \tau \,.\, M \mid M\, M \mid \mathbf{fix}(M) \end{array}$$

where $x \in \mathbb{V}$, an infinite set of variables.

Technicality: We identify expressions up to α-conversion of bound variables (created by the $\mathbf{fn}$ expression-former): by definition a PCF term is an α-equivalence class of expressions.

PCF typing relation, $\Gamma \vdash M : \tau$

- Γ is a type environment, *i.e.* a finite partial function mapping variables to types (whose domain of definition is denoted $dom(\Gamma)$)

  $x_1 : \tau_1, x_2 : \tau_2, \ldots, x_n : \tau_n$

- M is a term

- τ is a type.

PCF typing relation, $\Gamma \vdash M : \tau$

- Γ is a type environment, *i.e.* a finite partial function mapping variables to types (whose domain of definition is denoted $dom(\Gamma)$)
- M is a term
- τ is a type.

Notation:

$M : \tau$ means M is closed and $\emptyset \vdash M : \tau$ holds.

$\mathrm{PCF}_\tau \stackrel{\text{def}}{=} \{M \mid M : \tau\}$.

PCF typing relation (sample rules)

$$(:_{\mathrm{fn}}) \quad \frac{\Gamma[x \mapsto \tau] \vdash M : \tau'}{\Gamma \vdash \mathbf{fn}\, x : \tau \,.\, M : \tau \to \tau'} \quad \text{if } x \notin dom(\Gamma)$$

PCF typing relation (sample rules)

$$(:_{\text{fn}}) \quad \frac{\Gamma[x \mapsto \tau] \vdash M : \tau'}{\Gamma \vdash \mathbf{fn}\, x : \tau \,.\, M : \tau \to \tau'} \quad \text{if } x \notin dom(\Gamma)$$

$$(:_{\text{app}}) \quad \frac{\Gamma \vdash M_1 : \tau \to \tau' \quad \Gamma \vdash M_2 : \tau}{\Gamma \vdash M_1\, M_2 : \tau'}$$

PCF typing relation (sample rules)

$$(:_{\mathrm{fn}}) \quad \frac{\Gamma[x \mapsto \tau] \vdash M : \tau'}{\Gamma \vdash \mathbf{fn}\, x : \tau \,.\, M : \tau \to \tau'} \quad \text{if } x \notin dom(\Gamma)$$

$$(:_{\mathrm{app}}) \quad \frac{\Gamma \vdash M_1 : \tau \to \tau' \quad \Gamma \vdash M_2 : \tau}{\Gamma \vdash M_1\, M_2 : \tau'}$$

$$(:_{\mathrm{fix}}) \quad \frac{\Gamma \vdash M : \tau \to \tau}{\Gamma \vdash \mathbf{fix}(M) : \tau}$$

$$f \in (D \to D) \implies fix(f) \in D$$