For a poset P and a monotone function $f : P \to P$ we define $\underline{fix}(f) \in P$

(1) $f(\underline{fix} f) \sqsubseteq \underline{fix}(f)$

(2) $f(x) \sqsubseteq x \Rightarrow \underline{fix}(f) \sqsubseteq x$

If $\underline{fix}(f)$ exists then it is unique.

Let $p, q \in P$ satisfy (1) & (2). Then $p = q$.

By (1) for p we have $f(p) \sqsubseteq p$.

By (2) for q we have $q \sqsubseteq p$.

$\boxed{?}$ Do all monotone functions on posets have a least pre-fixed point?

$\boxed{!}$ No. (E.g. $\neg : \mathrm{Bool} \to \mathrm{Bool}$)

Least pre-fixed points are fixed points

If it exists, the least pre-fixed point of a mononote function on a partial order is necessarily a fixed point.

I.e. $f(\underline{fix}(f)) = \underline{fix}(f)$

We show: $f(\underline{fix}\, f) \sqsubseteq \underline{fix}(f)$ by (lfp 1)

$\boxed{?}$ $\underline{fix}(f) \sqsubseteq f(\underline{fix}(f))$?

$$\frac{x \sqsubseteq y}{fx \sqsubseteq fy}$$

since f is monotone

$$\frac{}{f(\mathrm{fix}\, f) \sqsubseteq \mathrm{fix}(f)}\ (lfp1)$$

$f(\mathrm{fix}\, f)$ is a prefixed point

i.e:

$$\frac{f(f(\mathrm{fix}\, f)) \sqsubseteq f(\mathrm{fix}\, f)}{\mathrm{fix}(f) \sqsubseteq f(\mathrm{fix}\, f)}\ (lfp2)$$

Thesis*

All domains of computation are
complete partial orders with a least element.

passage to the limit.

Thesis*

All domains of computation are complete partial orders with a least element.

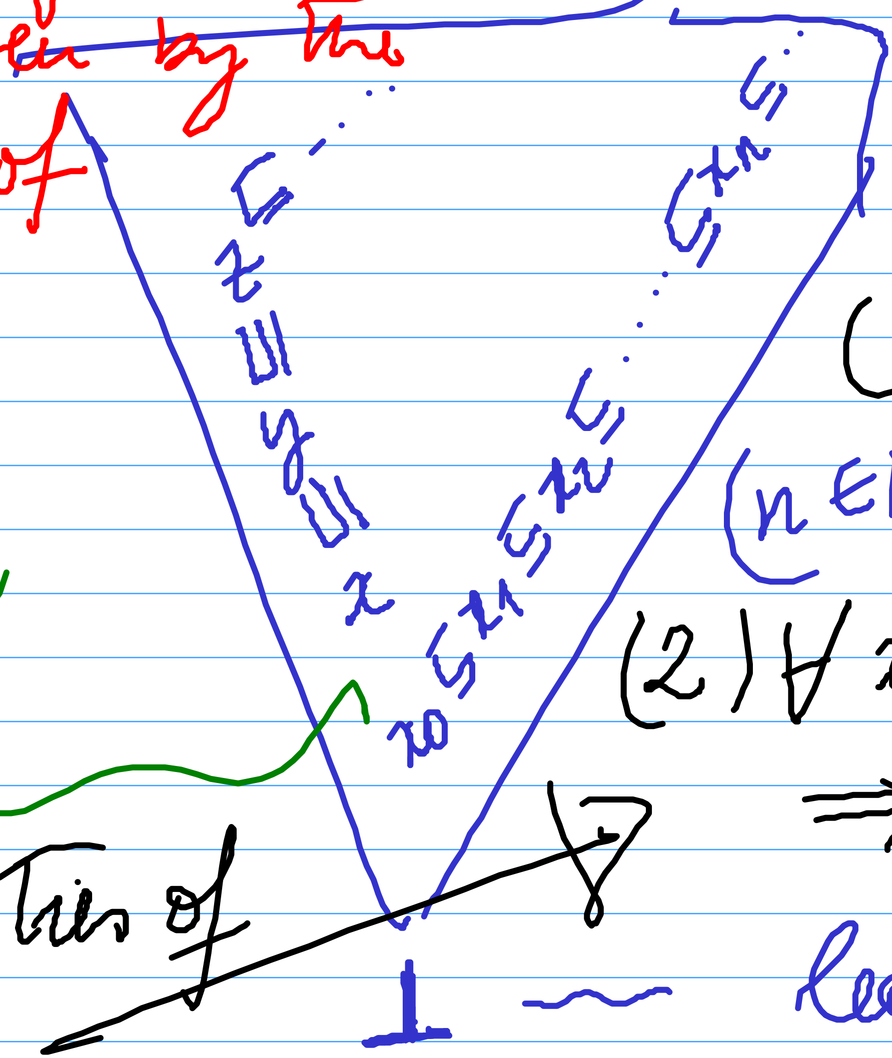
All computable functions are continuous.

monotone + lub preservation property.

Domain

sums up all the information given by the chain of the x_n $\sim \exists \bigsqcup_n (x_n)$

is a partial order (of information)



an ω-chain

(1) $\forall i.\ x_i \sqsubseteq \bigsqcup_n x_n$ $(n \in \mathbb{N})$

(2) $\forall x.\ \forall i\ x_i \sqsubseteq x \Rightarrow \bigsqcup_n x_n \sqsubseteq x$

The properties of lubs

$\bot$ ~ least element

Cpo's and domains

A chain complete poset, or cpo for short, is a poset $(D, \sqsubseteq)$ in which all countable increasing chains $d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \ldots$ have least upper bounds, $\bigsqcup_{n \geq 0} d_n$:

$$\forall m \geq 0 \,.\, d_m \sqsubseteq \bigsqcup_{n \geq 0} d_n \qquad \text{(lub1)}$$

$$\forall d \in D \,.\, (\forall m \geq 0 \,.\, d_m \sqsubseteq d) \;\Rightarrow\; \bigsqcup_{n \geq 0} d_n \sqsubseteq d. \qquad \text{(lub2)}$$

A domain is a cpo that possesses a least element, $\perp$:

$$\forall d \in D \,.\, \perp \sqsubseteq d.$$

Examples:

(1) Are finite posets domains?

$(\{true, false\}, =)$ $\qquad$ $\bullet$ true $\quad$ $\bullet$ false

(2) Are all finite posets with least element domains?

Yes, because every chain in it looks like

$$x_0 \sqsubseteq x_1 \sqsubseteq x_2 \sqsubseteq \cdots \sqsubseteq x_n \sqsubseteq x_n \sqsubseteq x_n \sqsubseteq \cdots$$

which has lub x_n.

(3) $(\mathbb{N}, \leq)$ is not a domain.

because the chain

$$0 \leq 1 \leq 2 \leq \cdots \leq n \leq \cdots \qquad (n \in \mathbb{N})$$

does not have a lub, in fact it does not have any upper bound.

(4) Define Ω to have underlying set

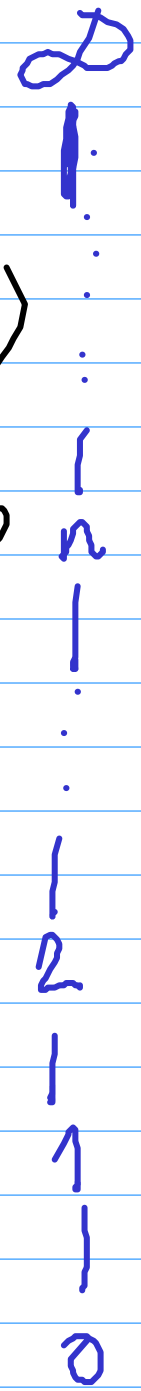
$$\mathbb{N} \uplus \{\infty\}$$

and order

$$n \sqsubseteq m \quad \forall\, n \leq m \text{ in } \mathbb{N}$$

and

$$n \sqsubseteq \infty \quad \forall\, n$$



(5) $(\mathcal{P}(X), \subseteq)$

$=$

$\{ S \mid S \text{ is a subset of } X \}$

is a domain with $\bot = \emptyset$

and lubs given by unions:

for $S_0 \subseteq S_1 \subseteq S_2 \subseteq \cdots \subseteq S_n \subseteq \cdots$

the lub is

$$\bigcup_n S_n$$

$$\frac{}{\perp \sqsubseteq x}$$

$$\frac{}{x_i \sqsubseteq \bigsqcup_{n \geq 0} x_n} \quad (i \geq 0 \text{ and } \langle x_n \rangle \text{ a chain})$$

$$\frac{\forall n \geq 0 \,.\, x_n \sqsubseteq x}{\bigsqcup_{n \geq 0} x_n \sqsubseteq x} \quad (\langle x_i \rangle \text{ a chain})$$

Domain of partial functions, $X \rightharpoonup Y$

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Underlying set: all partial functions, f, with domain of definition $dom(f) \subseteq X$ and taking values in Y.

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Partial order:

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\begin{array}{lll}
f \sqsubseteq g & \text{iff} & dom(f) \subseteq dom(g) \text{ and} \\
& & \forall x \in dom(f).\ f(x) = g(x) \\
& \text{iff} & graph(f) \subseteq graph(g)
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Lub of chain $f_0 \sqsubseteq f_1 \sqsubseteq f_2 \sqsubseteq \ldots$ is the partial function f with $dom(f) = \bigcup_{n \geq 0} dom(f_n)$ and

$$
f(x) = \begin{cases} f_n(x) & \text{if } x \in dom(f_n) \text{, some } n \\ \text{undefined} & \text{otherwise} \end{cases}
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$\underline{graph}\left(\bigsqcup_n f_n\right) = \bigcup_n \underline{graph}(f_n)$

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$$

Least element $\perp$ is the totally undefined partial function.

$d \sqsubseteq d \sqsubseteq d \sqsubseteq \cdots \sqsubseteq d \sqsubseteq \cdots$

Some properties of lubs of chains

$\forall n \; d \sqsubseteq d$

Let D be a cpo.

$\bigsqcup_n d \sqsubseteq d$ $\quad$ $d \sqsubseteq \bigsqcup_n d$

1. For $d \in D$, $\bigsqcup_n d = d$.

2. For every chain $d_0 \sqsubseteq d_1 \sqsubseteq \ldots \sqsubseteq d_n \sqsubseteq \ldots$ in D, $\bigsqcup_n d_n$

$$\bigsqcup_n d_n = \bigsqcup_n d_{N+n}$$

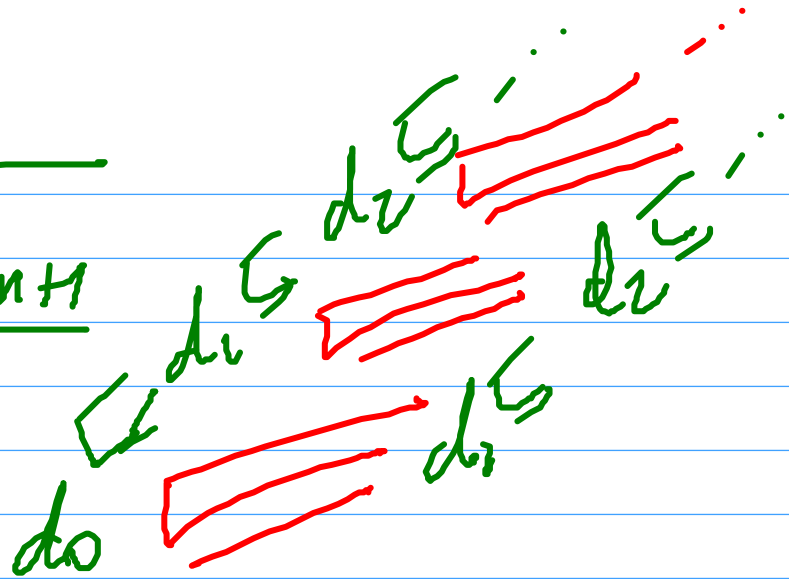
for all $N \in \mathbb{N}$.

$d_N \sqsubseteq d_{N+1} \sqsubseteq \cdots \sqsubseteq d_{N+n} \sqsubseteq \cdots$

$\bigsqcup_n d_{N+n}$

$$\boxed{\bigsqcup_n d_n \stackrel{?}{=} \bigsqcup_n d_{n+1}}$$

$$\frac{(i=0)\ d_0 \sqsubseteq d_1 \sqsubseteq \bigsqcup_n d_{n+1} \qquad (i \geq 1)\ \overline{d_i \sqsubseteq \bigsqcup_n d_{n+1}}}{\dfrac{\forall i \geq 0.\ d_i \sqsubseteq \bigsqcup_n d_{n+1}}{\bigsqcup_n d_n \sqsubseteq \bigsqcup_n d_{n+1}}}$$



$$\frac{\forall i \geq 1 \quad \overline{d_i \sqsubseteq \bigsqcup_n d_n}}{\bigsqcup_n d_{n+1} \sqsubseteq \bigsqcup_n d_n}$$

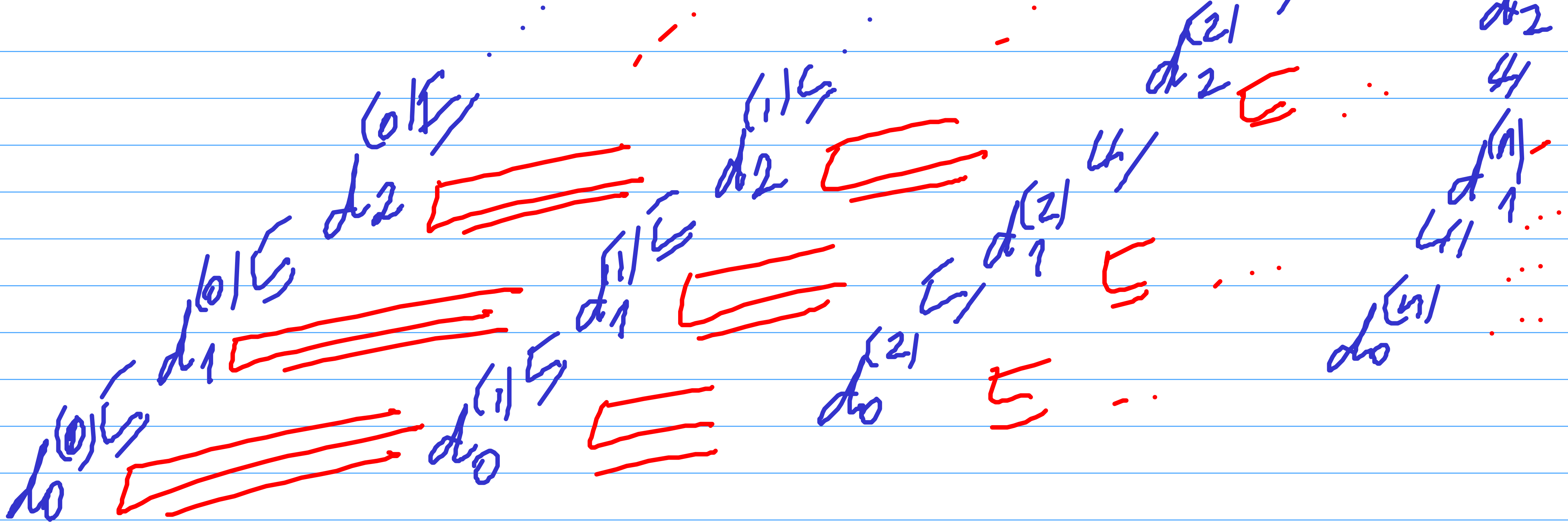
$$
\begin{tikzcd}[column sep=small, row sep=small]
 & & \cdots & & \cdots \\
 & d_2 \arrow[ur, phantom, "\sqsubseteq"] \arrow[rr, phantom, "\sqsubseteq"] & & e_2 \arrow[ur, phantom, "\sqsubseteq"] & \\
 d_1 \arrow[ur, phantom, "\sqsubseteq"] \arrow[rr, phantom, "\sqsubseteq"] & & e_1 \arrow[ur, phantom, "\sqsubseteq"] & & \\
 d_0 \arrow[u, phantom, "\sqsubseteq"] \arrow[rr, phantom, "\sqsubseteq"] & & e_0 \arrow[u, phantom, "\sqsubseteq"] & &
\end{tikzcd}
$$

Prop

$$\bigsqcup_n d_n \sqsubseteq \bigsqcup_n e_n$$

$$
\dfrac{\dfrac{\overline{d_i \sqsubseteq e_i} \qquad \overline{e_i \sqsubseteq \bigsqcup_n e_n}}{\forall i.\ d_i \sqsubseteq \bigsqcup_n e_n}}{\bigsqcup_n d_n \sqsubseteq \bigsqcup_n e_n}
$$

$$\bigsqcup_n d_n^{(0)} \sqsubseteq \bigsqcup_n d_n^{(1)} \sqsubseteq \bigsqcup_n d_n^{(2)} \sqsubseteq$$



Fix the superindex, say to m : $\bigsqcup_n d_n^{(m)}$

We can take $\bigsqcup_m \bigsqcup_n d_n^{(m)}$