Complexity Theory
Lecture 11

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Inclusions

We have the following inclusions:

$$\mathsf{L} \subseteq \mathsf{NL} \subseteq \mathsf{P} \subseteq \mathsf{NP} \subseteq \mathsf{PSPACE} \subseteq \mathsf{NPSPACE} \subseteq \mathsf{EXP}$$

where $\mathsf{EXP} = \bigcup_{k=1}^{\infty} \mathsf{TIME}(2^{n^k})$

Moreover,

$$\mathsf{L} \subseteq \mathsf{NL} \cap \mathsf{co\text{-}NL}$$

$$\mathsf{P} \subseteq \mathsf{NP} \cap \mathsf{co\text{-}NP}$$

$$\mathsf{PSPACE} \subseteq \mathsf{NPSPACE} \cap \mathsf{co\text{-}NPSPACE}$$

Establishing Inclusions

To establish the known inclusions between the main complexity classes, we prove the following.

- $\mathsf{SPACE}(f(n)) \subseteq \mathsf{NSPACE}(f(n))$;
- $\mathsf{TIME}(f(n)) \subseteq \mathsf{NTIME}(f(n))$;
- $\mathsf{NTIME}(f(n)) \subseteq \mathsf{SPACE}(f(n))$;
- $\mathsf{NSPACE}(f(n)) \subseteq \mathsf{TIME}(k^{\log n + f(n)})$;

The first two are straightforward from definitions.

The third is an easy simulation.

The last requires some more work.

Reachability

Recall the Reachability problem: given a *directed* graph $G = (V, E)$ and two nodes $a, b \in V$, determine whether there is a path from a to b in G.

A simple search algorithm solves it:

1. mark node a, leaving other nodes unmarked, and initialise set S to $\{a\}$;
2. while S is not empty, choose node i in S: remove i from S and for all j such that there is an edge (i, j) and j is unmarked, mark j and add j to S;
3. if b is marked, accept else reject.

NL Reachability

We can construct an algorithm to show that the Reachability problem is in NL:

1. write the index of node a in the work space;
2. if i is the index currently written on the work space:
   (a) if $i = b$ then accept, else
   guess an index j ($\log n$ bits) and write it on the work space.
   (b) if (i, j) is not an edge, reject, else replace i by j and return to (2).

We can use the $O(n^2)$ algorithm for Reachability to show that:

$\mathsf{NSPACE}(f(n)) \subseteq \mathsf{TIME}(k^{\log n + f(n)})$

for some constant k.

Let M be a nondeterministic machine working in space bounds $f(n)$.

For any input x of length n, there is a constant c (depending on the number of states and alphabet of M) such that the total number of possible configurations of M within space bounds $f(n)$ is bounded by $n \cdot c^{f(n)}$.

> Here, $c^{f(n)}$ represents the number of different possible contents of the work space, and n different head positions on the input.

Configuration Graph

Define the *configuration graph* of M, x to be the graph whose nodes are the possible configurations, and there is an edge from i to j if, and only if, $i \rightarrow_M j$.

Then, M accepts x if, and only if, some accepting configuration is reachable from the starting configuration $(s, \triangleright, x, \triangleright, \varepsilon)$ in the configuration graph of M, x.

Using the $O(n^2)$ algorithm for Reachability, we get that $L(M)$—the language accepted by M—can be decided by a deterministic machine operating in time

$$c'(nc^{f(n)})^2 \sim c'c^{2(\log n + f(n))} \sim k^{(\log n + f(n))}$$

In particular, this establishes that $\mathsf{NL} \subseteq \mathsf{P}$ and $\mathsf{NPSPACE} \subseteq \mathsf{EXP}$.

Savitch's Theorem

Further simulation results for nondeterministic space are obtained by other algorithms for Reachability.

We can show that Reachability can be solved by a *deterministic* algorithm in $O((\log n)^2)$ space.

Consider the following recursive algorithm for determining whether there is a path from a to b of length at most i (for i a power of 2):

$O((\log n)^2)$ space Reachability algorithm:

Path(a, b, i)

if $i = 1$ and $a \neq b$ and (a, b) is not an edge reject
else if (a, b) is an edge or $a = b$ accept
else, for each node x, check:

1. is there a path $a - x$ of length $i/2$; and
2. is there a path $x - b$ of length $i/2$?

if such an x is found, then accept, else reject.

The maximum depth of recursion is $\log n$, and the number of bits of information kept at each stage is $3 \log n$.

Savitch's Theorem - 2

The space efficient algorithm for reachability used on the configuration graph of a nondeterministic machine shows:

$$\mathsf{NSPACE}(f(n)) \subseteq \mathsf{SPACE}(f(n)^2)$$

for $f(n) \geq \log n$.

This yields

$$\mathsf{PSPACE} = \mathsf{NPSPACE} = \text{co-}\mathsf{NPSPACE}.$$

Complementation

A still more clever algorithm for Reachability has been used to show that nondeterministic space classes are closed under complementation:

If $f(n) \geq \log n$, then

$$\mathsf{NSPACE}(f(n)) = \text{co-}\mathsf{NSPACE}(f(n))$$

In particular

$$\mathsf{NL} = \text{co-}\mathsf{NL}.$$

Logarithmic Space Reductions

We write

$$A \leq_L B$$

if there is a reduction f of A to B that is computable by a deterministic Turing machine using $O(\log n)$ workspace (with a *read-only* input tape and *write-only* output tape).

Note: We can compose $\leq_L$ reductions. So,

$$\text{if } A \leq_L B \text{ and } B \leq_L C \text{ then } A \leq_L C$$

NP-complete Problems

Analysing carefully the reductions we constructed in our proofs of NP-completeness, we can see that SAT and the various other NP-complete problems are actually complete under $\leq_L$ reductions.

Thus, if $\mathsf{SAT} \leq_L A$ for some problem A in L then not only $\mathsf{P} = \mathsf{NP}$ but also $\mathsf{L} = \mathsf{NP}$.

P-complete Problems

It makes little sense to talk of complete problems for the class P with respect to polynomial time reducibility $\leq_P$.

There are problems that are complete for P with respect to *logarithmic space* reductions $\leq_L$.

One example is CVP—the circuit value problem.

- If $\mathsf{CVP} \in \mathsf{L}$ then $\mathsf{L} = \mathsf{P}$.
- If $\mathsf{CVP} \in \mathsf{NL}$ then $\mathsf{NL} = \mathsf{P}$.

Provable Intractability

Our aim now is to show that there are languages (*or, equivalently, decision problems*) that we can prove are not in P.

This is done by showing that, for every *reasonable* function f, there is a language that is not in $\mathsf{TIME}(f(n))$.

The proof is based on the diagonal method, as in the proof of the undecidability of the halting problem.

Constructible Functions

A complexity class such as $\mathsf{TIME}(f(n))$ can be very unnatural, if $f(n)$ is.

We restrict our bounding functions $f(n)$ to be proper functions:

Definition
A function $f : \mathbb{N} \to \mathbb{N}$ is *constructible* if:

- f is non-decreasing, i.e. $f(n+1) \geq f(n)$ for all n; and
- there is a deterministic machine M which, on any input of length n, replaces the input with the string $0^{f(n)}$, and M runs in time $O(n + f(n))$ and uses $O(f(n))$ *work space.*

Examples

All of the following functions are constructible:

- $\lceil \log n \rceil$;
- n^2;
- n;
- 2^n.

If f and g are constructible functions, then so are $f + g$, $f \cdot g$, 2^f and $f(g)$ (this last, provided that $f(n) > n$).

Using Constructible Functions

$\mathsf{NTIME}(f(n))$ can be defined as the class of those languages L accepted by a *nondeterministic* Turing machine M, such that for every $x \in L$, there is an accepting computation of M on x of length at most $O(f(n))$.

If f is a constructible function then any language in $\mathsf{NTIME}(f(n))$ is accepted by a machine for which all computations are of length at most $O(f(n))$.

Also, given a Turing machine M and a constructible function f, we can define a machine that simulates M for $f(n)$ steps.

Inclusions

The inclusions we proved between complexity classes:

- $\mathsf{NTIME}(f(n)) \subseteq \mathsf{SPACE}(f(n))$;
- $\mathsf{NSPACE}(f(n)) \subseteq \mathsf{TIME}(k^{\log n + f(n)})$;
- $\mathsf{NSPACE}(f(n)) \subseteq \mathsf{SPACE}(f(n)^2)$

really only work for *constructible* functions f.

The inclusions are established by showing that a deterministic machine can simulate a nondeterministic machine M for $f(n)$ steps.

For this, we have to be able to compute f within the required bounds.