Complexity Theory
Easter 2013
Suggested Exercises 2

1. Given a graph $G = (V, E)$, a set $U \subseteq V$ of vertices is called a *vertex cover* of G if, for each edge $(u, v) \in E$, either $u \in U$ or $v \in U$. That is, each edge has at least one end point in U. The decision problem V-COVER is defined as:

   given a graph $G = (V, E)$, and an integer K, does G contain a vertex cover with K or *fewer* elements?

   (a) Show a polynomial time reduction from IND to V-COVER.

   (b) Use (a) to argue that V-COVER is NP-complete.

2. The problem of four dimensional matching, 4DM, is defined analogously with 3DM:

   Given four sets, W, X, Y and Z, each with n elements, and a set of quadruples $M \subseteq W \times X \times Y \times Z$, is there a subset $M' \subseteq M$, such that each element of W, X, Y and Z appears in exactly one triple in M'.

   Show that 4DM is NP-complete.

3. Given a graph $G = (V, E)$, a *source vertex* $s \in V$ and a *target vertex* $t \in V$, a *Hamiltonian Path* from s to t in G is a path that begins at s, ends at t and visits every vertex in V exactly once. We define the decision problem HamPath as:

   given a graph $G = (V, E)$ and vertices $s, t \in V$, does G contain a Hamiltonian path from s to t?

   (a) Give a polynomial time reduction from the Hamiltonian cycle problem to HamPath.

   (b) Give a polynomial time reduction from HamPath to the problem of determining whether a graph has a Hamiltonian cycle.

   Hint: consider adding a vertex to the graph.

4. We know from the Cook-Levin theorem that every problem in NP is reducible to SAT. Sometimes it is easy to give an explicit reduction. In this exercise you are asked to give such explicit reductions for two graph problems: 3-Col and HAM. That is,

(a) describe how to obtain, for any graph $G = (V, E)$, a Boolean expression ϕ_G so that ϕ_G is satisfiable if, and only if, G is 3-colourable; and

(b) describe how to obtain, for any graph $G = (V, E)$, a Boolean expression ϕ_G so that ϕ_G is satisfiable if, and only if, G contains a Hamiltonian cycle.

5. We use $x; 0^n$ to denote the string that is obtained by concatenating the string x with a separator ; followed by n occurrences of 0. If $[M]$ represents the string encoding of a *non-deterministic* Turing machine M, show that the following language is NP-complete:

$$\{[M]; x; 0^n \mid M \text{ accepts } x \text{ within } n \text{ steps}\}.$$

Hint: rather than attempting a reduction from a particular NP-complete problem, it is easier to show this from first principles, i.e. construct a reduction for any NDTM M, and polynomial bound p.