M.Phil in Advanced Computer Science 2011–12

Module R07: Introductory Logic (AM)

Exercises

Many of these exercises explore around points discussed in lectures; and are perhaps slightly nastier than an exam question.
[The symbol'*' against a question shows a rather more open-ended question.]

Question 1

(a) Define the idea of well-formed formula (wff) for propositional logic. "Well-formed" historically meant that formulae (seen as strings) were properly parethesised. What does it mean when we treat formulae as trees?

The idea of well-formed formula was to distinguish strings *of characters which represent formulae from malformed strings such as* "$) \wedge (($". *It pre-dated computer science. Hence various syntactic requirements were imposed (such as fully-parenthesised formulae to avoid ambiguity for things like $A \wedge B \vee C$). These were then informally relaxed in examples by conventions. The theory of syntax and BNF in computer science can be used to define the concrete syntax of wffs including operator precedence. When formulae are seen as trees the notion of wff just means the tree is an instance of the abstract syntax tree for*

$$\phi ::= A \mid \phi \wedge \phi \mid \phi \vee \phi \mid etc$$

where A is a propositional variable (posibly itself having syntax as a sequence of characters and ϕ ranges over wffs. Note that here, as usual in BNF, repeated uses of ϕ can represent different wff instances.

(b) What is a valuation? Explain what it means for a wff to be: valid, satisfiable, unsatisfiable. If wff A is satisfiable then is $\neg A$ unsatisfiable?

A valuation is a function from propositional variables to booleans (true,false). (It need only be defined on propositional variables occurring in the formula of interest.)

A formula is valid if it evaluates to true (using standard truth tables for boolean connectives) for all valuations. It is sasifiable if we replace 'for all with 'some' and unsatisfiable if we replace 'true' with 'false'

(c) Explain the difference between $\models$ and $\vdash$.

$\models \phi$ means ϕ is valid as above. We generalise this to $\Gamma \models \phi$ for a set of wffs Γ to mean that, whenever a valuation makes all the wffs in Γ evaluate to true then it also makes ϕ evaluate to true.

By contrast $\Gamma \vdash_R \phi$ means ϕ may be syntactically derived ('proved' or 'deduced') from Γ using inference rules in R. We write $\vdash$ for $\vdash_R$ when the set of inference rules is clear from context (e.g. when we have established that R is sound and complete).

(d) Suppose we use two axiom schemes (where A, B, C stand for any wff and treating '$\rightarrow$' as right-associative): $A \rightarrow B \rightarrow A$ and $(A \rightarrow B \rightarrow C) \rightarrow (A \rightarrow B) \rightarrow (A \rightarrow C)$ together with modus ponens.

(i) Given any wff A, can $A \rightarrow A$ be deduced?

Yes. Let's write K for the first axiom, S for the second and I for $A \rightarrow A$. People familiar with functional programming will note that $SKK = I$ as combinators (and by Curry-Howard ...).

This is a short-cut to finding the deduction:

$$\frac{\dfrac{SK}{\cdots} \quad K}{I}$$

(ii*) Can you find a valid wff which is not deducible?

Yes, but tricky, e.g. DN (without which S and K would not be complete): $(\neg\phi \rightarrow \neg\psi) \rightarrow ((\neg\phi \rightarrow \psi) \rightarrow \phi)$

(iii) Can you find a wff which is deducible but not valid?

No – these axioms are sound. I.e. (inductively) both axioms are valid and modus ponens preserves validity.

(iv) Express this using words like 'soundness' and 'completeness'.

Axioms S and K together with MP are sound but not complete

Question 2

(a) Summarise the ideas of first-order logic, including wffs, interpretations, model.

Bookwork.

(b) Explain the notion of semantic entailment $\Gamma \models \phi$.

Bookwork – explain the notion of evaluating a wff w.r.t. an interpretation *(of function symbols and predicate symbols); and and the values (taken from the universe of discourse) of free individual variables.*

(c) Explain the difference between $\models$ and $\vdash$. To what extent are they identical or equivalent concepts

Bookwork – parallels the FOL version above. They are are not identical (one is 'truth' (validity) and one is provability. However, by Gödel's Completeness Theorem there is a sound and complete set of inference rules R such that $\Gamma \vdash_R \phi$ iff $\Gamma \models \phi$ which makes them equivalent as meta-logical relations between sets of wffs and wffs.

Question 3

(a) Explain the notions of compactness, a set of clauses being consistent and a set of clauses being satisfiable.

*C*ompactness has various equivalent formulations, but a simple one is "If every finite subset of Σ has a model then so does Σ."

One formulation of several: Σ is consistent if there is some wff ϕ for which $\Sigma \models \phi$ does not hold.

Σ is satisfiable if there is some interpretation which makes every element of Σ true.

(b) Prove compactness for first-order logic assuming there is a sound and complete set of axioms and inference rules.

Since consistency and satisfiability are equivalent statements for given soundness and completeness, compactness is equivalent to "if every finite subset of Σ is consistent then so is Σ". I.e. "if Σ is inconsistent then so is some finite subset". But this is necessarily true since a proof ($\vdash$) of inconsistency of Σ can only use a finite number of its elements as proofs are finite.

Question 4

(a) Explain what it means for a theory to be complete.

For every ϕ we have $\Gamma \vdash \phi$ or $\Gamma \vdash \neg\phi$.

(b) Does a theory being complete mean that all its models are isomorphic? Give reasons.

No: Skölem-Lowenheim says that if a theory has an infinite model then it has models of all cardinalities, which therefore lack bijections (and hence isomorphisms).

(c) Can a theory have exactly two models – one infinte one and one finite one?

Rather mis-formed question – previous question shows "one infinite model implies an infinte number of distinct models". However we could have a Peano-number-like system which replaced $\forall x\ \neg(S(x) = 0)$ with $(\forall x\ \neg(S(x) = 0)) \vee (\forall x\ (S(x) = 0))$ which has a single finite model (with one element) and one countably infinite model

(the Peano numbers)

(d) To what extent can one write axioms which have models exactly when the model has an odd number of elements.

Imperfectly! If a theory has arbitrarily large finite models then it has an infinite model (and we don't generally regard infinity as an odd number!).

However, we can write an axiom A_k meaning "does not have k elements", e.g. for $k = 3$ we have

$$\neg\exists x\exists y\exists z(x \neq y \wedge y \neq z \wedge x \neq x \wedge \forall t(t = x \vee t = y \vee t = z))$$

The set of axioms $\{A_2, A_4, A_6, \ldots\}$ then finitely axiomatises all odd models (by forbidding even models) together with various infinite models.

Question 5*

Attempt to axiomatise set theory. You should define a binary relation which models '$\in$', but prefer to define other relations such as '$\subseteq$' as abbreviations for wffs involving '$\in$'.

This is rather open-ended. I'd start with a binary relation $\in$ of arity two, and take inspiration from the Peano axiomatisation of numbers. One could either add $\emptyset$ as a constant (function or arity zero) or axiomatise if as e by: $\exists e\ \forall x\ \neg(x \in e)$. The site `http://www.mtnmath.com/whatth/node23.html` *gives the standard axioms with explanation, but the sort of thing one needs is:*

- $\forall S \forall T\ ((\forall x\ (x \in S \leftrightarrow x \in T)) \leftrightarrow S = T)$

Then then $S \subseteq T$ can be seen as an abbreviation of $\forall x\ (x \in S \rightarrow x \in T)$ etc.

Question 6

(a) Explain the notion of deductive closure $Con\ \Gamma$ of a set of wffs Γ.

$Con\ \Gamma = \{\phi \mid (\Gamma \vdash \phi)\}$

(b*) Author X defines a theory Θ to be axiomatisable when there is a decidable (a.k.a. recursive) set of wffs Γ such that $\Theta = Con\ \Gamma$, while author Y defines it to be axiomatisable when there is a recursively enumerable set of wffs Γ such that $\Theta = Con\ \Gamma$. Which, if either, author is more generous?

These definitions are equivalent, but it's not obvious at first. Obviously "recursively enumerable" is more generous than "recursive". The question is "whether we can tell when an axiom is not in Γ" and this is unclear when Γ is only recursively *enumerable. But we can replace* $\Gamma = \{\gamma_1, \gamma_2, \gamma_3, \ldots\}$ *with* $\Gamma' = \{\gamma_1, \gamma_1 \wedge \gamma_2, \gamma_1 \wedge \gamma_2 \wedge \gamma_3, \ldots\}$*.* Γ *and* Γ' *have the same models, and* Γ' *is recursive (not merely recursively enumerable) because given a* ϕ *we can determine whether or not* ϕ *is in* Γ' *just checking members as enumerated until one appears which is bigger (in terms of size – number of characters in it) then* ϕ*.*

(c) Why do we not define a theory to be axiomatisable if it simply has a countable set of wffs Γ such that $\Theta = Con\ \Gamma$?

Because the idea of countability is too loose; we need some effective way of determining which things are true or false. For example if we enumerate Turing machines also wffs, but take Γ *to be those wffs which correspond to non-terminating Turing machines. Then we can't even mechanically decide what axioms the system has! (Of course, we can have philosphical discussion here about whether people can do non-effective things, but that's another subject.*