L108: Categorical theory and logic
Exercise sheet 2

Sam Staton
sam.staton@cl.cam.ac.uk

Revised: November 3, 2011, 12:00.

1. Recall the arrow category $\hat{\Sigma}$: the objects are functions, and the morphisms are commuting squares. Does the arrow category have finite products?

2. A *preorder* is a category with at most one morphism between any two objects. To give a preorder is to give a set X (of objects) and a relation $(\lesssim) \subseteq X \times X$ that is reflexive and transitive. Informally "$x \lesssim y$" means "there is a morphism $x \to y$"; reflexivity supplies the identity morphisms and transitivity provides composition.

   Do the following preorders have finite products?

   (a) The preorder $(\mathbb{N}, \leq)$ of natural numbers, with $\leq$ the usual "less than or equal" relation.

   (b) The preorder $(\mathcal{O}, \subseteq)$ where $\mathcal{O}$ is the set of open intervals of the real line and $\subseteq$ is set inclusion. (For illustration, consider the intervals $(-2, e) = \{x \in \mathbb{R} \mid -2 < x < e\}$ and $(-\infty, \pi) = \{x \in \mathbb{R} \mid x < \pi\}$; then $(-2, e) \subseteq (-\infty, \pi)$.)

3. Every natural number n can be understood as a set with n elements. Precisely, we understand the natural number n as the set $\{1, \ldots, n\}$. In particular, zero is understood as the empty set.

   Let $\mathbb{F}$ be the full subcategory of the category of sets whose objects are natural numbers. That is, the objects of $\mathbb{F}$ are natural numbers, considered as sets, and the morphisms are functions between the sets. Identities are the identity functions and composition is just composition of functions.

   Show that the category $\mathbb{F}$ has finite products.

4. Let the category $\mathbb{F}^{\mathrm{op}}$ be the opposite of $\mathbb{F}$. That is, the objects are natural numbers, but the morphisms go in the opposite direction. So a morphism $m \to n$ in $\mathbb{F}^{\mathrm{op}}$ is a function $n \to m$. Composition is still composition of functions. To be precise, given morphisms $f : n_1 \to n_2$ and $g : n_2 \to n_3$, i.e. functions $\bar{f} : n_2 \to n_1$ and $\bar{g} : n_3 \to n_2$, let the composite morphism $(g \circ f) : n_1 \to n_3$ be the composed function $(\bar{f} \circ \bar{g}) : n_3 \to n_1$.

   Show that the category $\mathbb{F}^{\mathrm{op}}$ has finite products.

5. Show that $\mathbb{F}^{\mathrm{op}}$ is a free category with finite products over one base type. What is the relationship between $\mathbb{F}^{\mathrm{op}}$ and the syntactic category for the type theory of products ($\mathbf{Cl}_\times$), if there is only one base type?