L108: Category theory and logic
Exercise sheet 1

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1. How many categories are there with two objects and four morphisms (up-to isomorphism of categories)?

2. Consider the category **CAT** whose objects are categories and whose morphisms are functors. Does this category have a terminal object?

3. Consider the monoid $(\mathbb{N}, +, 0)$ as a category with one object. (The morphisms are called natural numbers, and the composition of m with n is $m+n$, and the identity is 0.) Does this category have a terminal object?

4. **Definition.** An *initial object* in a category is an object X such that for any object Y there is a unique morphism $X \to Y$.

   (a) Does the category of sets have any initial objects?

   (b) Does the category of monoids have any initial objects?

   (c) State and prove a theorem making precise the following informal statement: "initial objects are unique up-to unique isomorphism".

5. Recall that the category Σ has two objects and one morphism between them, and $\mathbf{1}$ has one object and only an identity map. How many functors are there from Σ to $\mathbf{1}$? How many from $\mathbf{1}$ to Σ? How many natural transformations between them?

6. Recall the category of arrows, $\hat{\Sigma}$: the objects are functions between sets, and the morphisms are commuting squares. What functors can you find between the category of arrows and the category of sets (in either direction)? What natural transformations can you find between the functors?

7. Recall the free monoid functor, *List* : **Set** $\to$ **Monoid**. Describe another functor, called P : **Set** $\to$ **Monoid**, which acts on objects as follows: a set X is taken to the monoid $(\mathcal{P}(X), \cup, \emptyset)$, where $\mathcal{P}(X)$ is the powerset of X. Describe two natural transformations between these functors: *List* $\Rightarrow P$.

8. Recall the definition of a free monoid on a set. State and prove a theorem making precise the following statement: "free monoids are unique up-to unique isomorphism".

9. Let X be a set. Recall that a "monoid over X" is a structure (Y, m, e, f) where (Y, m, e) is a monoid and $f : X \to Y$ is a function. Describe a category whose objects are monoids over X. Does this category have an initial object?