Relating Denotational & Operational Semantics

[p79 et seq.]

PCF denotational semantics — aims

- PCF types $\tau \;\mapsto\;$ domains $[\![\tau]\!]$.
- Closed PCF terms $M : \tau \;\mapsto\;$ elements $[\![M]\!] \in [\![\tau]\!]$.

  Denotations of open terms will be continuous functions.
- Compositionality.

  In particular: $[\![M]\!] = [\![M']\!] \;\Rightarrow\; [\![\mathcal{C}[M]]\!] = [\![\mathcal{C}[M']]\!]$.
- Soundness.

  For any type τ, $M \Downarrow_\tau V \;\Rightarrow\; [\![M]\!] = [\![V]\!]$.
- Adequacy.

  For $\tau = bool$ or nat, $[\![M]\!] = [\![V]\!] \in [\![\tau]\!] \implies M \Downarrow_\tau V$.

Theorem. *For all types* τ *and closed terms* $M_1, M_2 \in \mathrm{PCF}_\tau$, *if* $[\![M_1]\!]$ *and* $[\![M_2]\!]$ *are equal elements of the domain* $[\![\tau]\!]$, *then* $M_1 \cong_{\mathrm{ctx}} M_2 : \tau$.

Proof.

$$\mathcal{C}[M_1] \Downarrow_{nat} V \Rightarrow [\![\mathcal{C}[M_1]]\!] = [\![V]\!] \quad \text{(soundness)}$$

$$\Rightarrow [\![\mathcal{C}[M_2]]\!] = [\![V]\!] \quad \text{(compositionality on } [\![M_1]\!] = [\![M_2]\!]\text{)}$$

$$\Rightarrow \mathcal{C}[M_2] \Downarrow_{nat} V \quad \text{(adequacy)}$$

and symmetrically (& similarly for $\Downarrow_{bool}$). $\square$

Compositionality

Proposition. *For all typing judgements* $\Gamma \vdash M : \tau$ *and* $\Gamma \vdash M' : \tau$, *and all contexts* $\mathcal{C}[-]$ *such that* $\Gamma' \vdash \mathcal{C}[M] : \tau'$ *and* $\Gamma' \vdash \mathcal{C}[M'] : \tau'$,

if $[\![\Gamma \vdash M]\!] = [\![\Gamma \vdash M']\!] : [\![\Gamma]\!] \to [\![\tau]\!]$

then $[\![\Gamma' \vdash \mathcal{C}[M]]\!] = [\![\Gamma' \vdash \mathcal{C}[M']]\!] : [\![\Gamma']\!] \to [\![\tau']\!]$

Proof is by induction on the structure of $\mathcal{C}$ — straightforward, given how $[\![-]\!]$ was defined.

E.g. if $\begin{cases} [\![M_1]\!] = [\![M_1']\!] \in [\![\tau \to \tau']\!] \\ [\![M_2]\!] = [\![M_2']\!] \in [\![\tau]\!] \end{cases}$

then

$$[\![M_1 M_2]\!] = ev \circ \langle [\![M_1]\!], [\![M_2]\!] \rangle$$

$$= ev \circ \langle [\![M_1']\!], [\![M_2']\!] \rangle$$

$$= [\![M_1' M_2']\!] .$$

Soundness

Proposition. *For all closed terms* $M, V \in \mathrm{PCF}_\tau$,

if $M \Downarrow_\tau V$ *then* $[\![M]\!] = [\![V]\!] \in [\![\tau]\!]$.

Proof: by rule induction for $M \Downarrow_\tau V$

Induction step for $(\Downarrow_{fix})$ $\dfrac{M\ fix(M) \Downarrow_\tau V}{fix(M) \Downarrow_\tau V}$

Have to show: $[\![M\ fix(M)]\!] = [\![V]\!] \Rightarrow [\![fix(M)]\!] = [\![V]\!]$

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Have to show: $[\![M\ fix(M)]\!] = [\![V]\!] \Rightarrow [\![fix(M)]\!] = [\![V]\!]$

But

$$[\![M\ fix(M)]\!] = [\![M]\!]\left([\![fix\,M]\!]\right)$$

$$= [\![M]\!]\left(fix([\![M]\!])\right)$$

by definition of $[\![-]\!]$

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$$[\![M\ fix(M)]\!] = [\![M]\!]([\![fix\ M]\!])$$

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$fix(f)$ is a fixed point of f

QED

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$$\begin{aligned}[\![M\ fix(M)]\!] &= [\![M]\!]([\![fix\ M]\!]) \\ &= [\![M]\!](fix([\![M]\!])) \\ &= fix([\![M]\!]) \\ &= [\![fix(M)]\!]\end{aligned}$$

by definition of $[\![-]\!]$

$fix(f)$ is a fixed point of f

QED

Induction step for $(\Downarrow_{cbn})$ $\dfrac{M_1 \Downarrow_{\tau\to\tau'} \mathbf{fn}\, x : \tau\, M \qquad M[M_2/x] \Downarrow_{\tau'} V}{M_1\, M_2 \Downarrow_{\tau'} V}$

Suppose $\begin{cases} [\![M_1]\!] = [\![\mathbf{fn}\, x : \tau .\, M]\!] \\ [\![M[M_2/x]]\!] = [\![V]\!] \end{cases}$

Have to prove $[\![M_1 M_2]\!] = [\![V]\!]$.

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$$\begin{aligned} [\![M_1 M_2]\!] &= [\![M_1]\!]([\![M_2]\!]) \\ &= [\![\mathbf{fn}\, x : \tau .\, M]\!]([\![M_2]\!]) \\ &= [\![\{x \mapsto \tau\} \vdash M]\!]([\![M_2]\!]) \end{aligned}$$

by definition of $[\![-]\!]$

Substitution property

Proposition. *Suppose that* $\Gamma \vdash M : \tau$ *and that* $\Gamma[x \mapsto \tau] \vdash M' : \tau'$, *so that we also have* $\Gamma \vdash M'[M/x] : \tau'$.

Then,

(ρ)

$$
\begin{aligned}
&[\![\Gamma \vdash M'[M/x]]\!](\rho) \\
&\quad = [\![\Gamma[x \mapsto \tau] \vdash M']\!]\big(\rho[x \mapsto [\![\Gamma \vdash M]\!]]\big)
\end{aligned}
$$

for all $\rho \in [\![\Gamma]\!]$.

(Can be proved by induction on the structure of the PCF expression M'.)

Substitution property

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\end{aligned}
$$

for all $\rho \in [\![\Gamma]\!]$*.*

In particular when $\Gamma = \emptyset$, $[\![\{x \mapsto \tau\} \vdash M']\!] : [\![\tau]\!] \to [\![\tau']\!]$ and

$$
\boxed{[\![M'[M/x]]\!] = [\![\{x \mapsto \tau\} \vdash M']\!]([\![M]\!])}
$$

Induction step for $(\Downarrow_{cbn})$ $\dfrac{M_1 \Downarrow_{\tau\to\tau'} \mathbf{fn}\, x:\tau\, M \qquad M[M_2/x] \Downarrow_{\tau'} V}{M_1 M_2 \Downarrow_{\tau'} V}$

Suppose $\begin{cases} [\![M_1]\!] = [\![\mathbf{fn}\, x:\tau.\, M]\!] \\ [\![M[M_2/x]]\!] = [\![V]\!] \end{cases}$

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Induction step for $(\Downarrow_{\text{cbn}})$ $\dfrac{M_1 \Downarrow_{\tau\to\tau'} \mathbf{fn}\, x : \tau\, M \qquad M[M_2/x] \Downarrow_{\tau'} V}{M_1\, M_2 \Downarrow_{\tau'} V}$

Suppose $\begin{cases} [\![M_1]\!] = [\![\mathbf{fn}\, x : \tau .\, M]\!] \\ [\![M[M_2/x]]\!] = [\![V]\!] \end{cases}$

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&= [\![\{x \mapsto \tau\} \vdash M]\!]([\![M_2]\!]) \\
&= [\![M[M_2/x]]\!] = [\![V]\!]
\end{aligned}$$

Q.E.D.

Adequacy

For any closed PCF terms M and V of *ground* type $\gamma \in \{nat, bool\}$ with V a value

$$[\![M]\!] = [\![V]\!] \in [\![\gamma]\!] \implies M \Downarrow_{\gamma} V \, .$$

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$$[\![\mathbf{fn}\, x : \tau.\, (\mathbf{fn}\, y : \tau.\, y)\, x]\!] \quad = \quad [\![\mathbf{fn}\, x : \tau.\, x]\!] \quad : [\![\tau]\!] \to [\![\tau]\!]$$

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but

$$\mathbf{fn}\, x : \tau.\, (\mathbf{fn}\, y : \tau.\, y)\, x \not\Downarrow_{\tau \to \tau} \mathbf{fn}\, x : \tau.\, x$$

Adequacy proof idea

1. We cannot proceed to prove the adequacy statement by a straightforward induction on the structure of terms.

   - Consider M to be $M_1\,M_2$, $\mathbf{fix}(M')$.

of type
nat
or
bool

M_1 & M'
are of function type

Adequacy proof idea

1. We cannot proceed to prove the adequacy statement by a straightforward induction on the structure of terms.

   ► Consider M to be $M_1\, M_2$, $\mathbf{fix}(M')$.

2. So we proceed to prove a stronger statement that applies to terms of arbitrary types and implies adequacy.

Adequacy proof idea

1. We cannot proceed to prove the adequacy statement by a straightforward induction on the structure of terms.

   ► Consider M to be $M_1\, M_2$, $\mathbf{fix}(M')$.

2. So we proceed to prove a stronger statement that applies to terms of arbitrary types and implies adequacy.

   This statement roughly takes the form:

   $$\boxed{[\![M]\!] \lhd_\tau M \text{ for all types } \tau \text{ and all } M \in \mathrm{PCF}_\tau}$$

   where the *formal approximation relations*

   $$\lhd_\tau \subseteq [\![\tau]\!] \times \mathrm{PCF}_\tau$$

   closed PCF terms of type τ

   are *logically* chosen to allow a proof by induction.

Requirements on the formal approximation relations, I

We want that, for $\gamma \in \{nat, bool\}$,

$$[\![M]\!] \lhd_\gamma M \text{ implies } \underbrace{\forall V\, ([\![M]\!] = [\![V]\!] \implies M \Downarrow_\gamma V)}_{\text{adequacy}}$$

Definition of $d \lhd_\gamma M \ \ (d \in [\![\gamma]\!], M \in \mathrm{PCF}_\gamma)$
for $\gamma \in \{nat, bool\}$

$$n \lhd_{nat} M \quad \stackrel{\text{def}}{\Leftrightarrow} \quad \big(n \in \mathbb{N} \ \Rightarrow\ M \Downarrow_{nat} \mathbf{succ}^n(\mathbf{0})\big)$$

$$\begin{aligned} b \lhd_{bool} M \quad \stackrel{\text{def}}{\Leftrightarrow} \quad & (b = true \ \Rightarrow\ M \Downarrow_{bool} \mathbf{true}) \\ & \& \ (b = false \ \Rightarrow\ M \Downarrow_{bool} \mathbf{false}) \end{aligned}$$

Proof of: $[\![M]\!] \lhd_\gamma M$ implies **adequacy**

Case $\gamma = nat$.

$$
\begin{aligned}
& [\![M]\!] = [\![V]\!] \\
& \quad \Longrightarrow \; [\![M]\!] = [\![\mathbf{succ}^n(\mathbf{0})]\!] && \text{for some } n \in \mathbb{N} \\
& \quad \Longrightarrow \; n = [\![M]\!] \lhd_\gamma M \\
& \quad \Longrightarrow \; M \Downarrow \mathbf{succ}^n(\mathbf{0}) && \text{by definition of } \lhd_{nat}
\end{aligned}
$$

Case $\gamma = bool$ is similar.

Requirements on the formal approximation relations, II

We want to be able to proceed by induction.

- Consider the case $M = M_1\, M_2$.

$\rightsquigarrow$ "*logical*" definition

relate functions ~~that~~ send related arguments to related results.

Definition of
$f \lhd_{\tau\to\tau'} M \ \big(f \in (\llbracket\tau\rrbracket \to \llbracket\tau'\rrbracket), M \in \mathrm{PCF}_{\tau\to\tau'}\big)$

Definition of $f \lhd_{\tau\to\tau'} M$ $\big(f \in (\llbracket\tau\rrbracket \to \llbracket\tau'\rrbracket), M \in \mathrm{PCF}_{\tau\to\tau'}\big)$

$$f \lhd_{\tau\to\tau'} M$$

$$\overset{\text{def}}{\Leftrightarrow} \quad \forall\, x \in \llbracket\tau\rrbracket, N \in \mathrm{PCF}_{\tau}$$

$$(x \lhd_{\tau} N \;\Rightarrow\; f(x) \lhd_{\tau'} M\, N)$$

The full

Definition of $d \lhd_\tau M$ ($d \in [\![\tau]\!], M \in \mathrm{PCF}_\tau$)

$$d \lhd_{nat} M \overset{\text{def}}{\Leftrightarrow} (d \in \mathbb{N} \Rightarrow M \Downarrow_{nat} \mathbf{succ}^d(\mathbf{0}))$$

$$\begin{aligned} d \lhd_{bool} M \overset{\text{def}}{\Leftrightarrow} & (d = \mathit{true} \Rightarrow M \Downarrow_{bool} \mathbf{true}) \\ & \& \ (d = \mathit{false} \Rightarrow M \Downarrow_{bool} \mathbf{false}) \end{aligned}$$

$$d \lhd_{\tau \to \tau'} M \overset{\text{def}}{\Leftrightarrow} \forall e, N \ (e \lhd_\tau N \Rightarrow d(e) \lhd_{\tau'} M\, N)$$

Fundamental property

Theorem. *For all* $\Gamma = \{x_1 \mapsto \tau_1, \ldots, x_n \mapsto \tau_n\}$ *and all* $\Gamma \vdash M : \tau$, *if* $d_1 \lhd_{\tau_1} M_1, \ldots, d_n \lhd_{\tau_n} M_n$ *then* $[\![\Gamma \vdash M]\!][x_1 \mapsto d_1, \ldots, x_n \mapsto d_n] \lhd_\tau M[M_1/x_1, \ldots, M_n/x_n]$.

Fundamental property

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NB. The case $\Gamma = \emptyset$ reduces to

$$[\![M]\!] \lhd_\tau M$$

for all $M \in \mathrm{PCF}_\tau$.

Requirements on the formal approximation relations, III

We want to be able to proceed by induction.

- Consider the case $M = \mathbf{fix}(M')$.

$\leadsto$ *admissibility* property

Admissibility property

Lemma. *For all types τ and $M \in \mathrm{PCF}_\tau$, the set*

$$\{ d \in [\![\tau]\!] \mid d \lhd_\tau M \}$$

is an admissible subset of $[\![\tau]\!]$.

(Easy proof by induction on structure of types τ.)

Further properties

Lemma. *For all types* τ, *elements* $d, d' \in [\![\tau]\!]$, *and terms* $M, N, V \in \mathrm{PCF}_\tau$,

1. *If* $d \sqsubseteq d'$ *and* $d' \lhd_\tau M$ *then* $d \lhd_\tau M$.
2. *If* $d \lhd_\tau M$ *and* $\forall V \, (M \Downarrow_\tau V \implies N \Downarrow_\tau V)$ *then* $d \lhd_\tau N$.

(Easy proofs by induction on structure of types τ.)

Fundamental property of the relations $\lhd_\tau$

Proposition. *If* $\Gamma \vdash M : \tau$ *is a valid PCF typing, then for all* Γ*-environments* ρ *and all* Γ*-substitutions* σ

$$\rho \lhd_\Gamma \sigma \;\Rightarrow\; [\![\Gamma \vdash M]\!](\rho) \lhd_\tau M[\sigma]$$

(Proof by rule induction for $\Gamma \vdash M : \tau$ — see p84-86)

- $\rho \lhd_\Gamma \sigma$ means that $\rho(x) \lhd_{\Gamma(x)} \sigma(x)$ holds for each $x \in dom(\Gamma)$.
- $M[\sigma]$ is the PCF term resulting from the simultaneous substitution of $\sigma(x)$ for x in M, each $x \in dom(\Gamma)$.

Proof of: $[\![M]\!] \lhd_\gamma M$ implies adequacy

Case $\gamma = nat$.

$$[\![M]\!] = [\![V]\!]$$

$$\implies [\![M]\!] = [\![\mathbf{succ}^n(\mathbf{0})]\!] \qquad \text{for some } n \in \mathbb{N}$$

$$\implies n = [\![M]\!] \lhd_\gamma M$$

$$\implies M \Downarrow \mathbf{succ}^n(\mathbf{0}) \qquad \text{by definition of } \lhd_{nat}$$

Case $\gamma = bool$ is similar.