# Complexity Theory
# Lecture 8

Anuj Dawar

University of Cambridge Computer Laboratory
Lent Term 2012

http://www.cl.cam.ac.uk/teaching/1112/Complexity/





## Responses to NP-Completeness

*Confronted by an NP-complete problem, say constructing a timetable, what can one do?*

- It's a single instance, does asymptotic complexity matter?
- What's the critical size? Is scalability important?
- Are there guaranteed restrictions on the input? Will a special purpose algorithm suffice?
- Will an approximate solution suffice? Are performance guarantees required?
- Are there useful heuristics that can constrain a search? Ways of ordering choices to control backtracking?





## Validity

We define VAL—the set of *valid* Boolean expressions—to be those Boolean expressions for which every assignment of truth values to variables yields an expression equivalent to `true`.

$$\phi \in \mathsf{VAL} \quad \Leftrightarrow \quad \neg\phi \notin \mathsf{SAT}$$

By an exhaustive search algorithm similar to the one for SAT, VAL is in $\mathsf{TIME}(n^2 2^n)$.

Is $\mathsf{VAL} \in \mathsf{NP}$?





## Validity

$\overline{\mathsf{VAL}} = \{\phi \mid \phi \notin \mathsf{VAL}\}$—the *complement* of VAL is in NP.

Guess a *falsifying* truth assignment and verify it.

Such an algorithm does not work for VAL.

In this case, we have to determine whether *every* truth assignment results in `true`—a requirement that does not sit as well with the definition of acceptance by a nondeterministic machine.

## Complementation

If we interchange accepting and rejecting states in a deterministic machine that accepts the language $L$, we get one that accepts $\overline{L}$.

If a language $L \in \mathsf{P}$, then also $\overline{L} \in \mathsf{P}$.

Complexity classes defined in terms of nondeterministic machine models are not necessarily closed under complementation of languages.

Define,

co-NP – the languages whose complements are in NP.





## Succinct Certificates

The complexity class NP can be characterised as the collection of languages of the form:

$$L = \{x \mid \exists y R(x, y)\}$$

Where $R$ is a relation on strings satisfying two key conditions

1. $R$ is decidable in polynomial time.
2. $R$ is *polynomially balanced*. That is, there is a polynomial $p$ such that if $R(x, y)$ and the length of $x$ is $n$, then the length of $y$ is no more than $p(n)$.





## Succinct Certificates

$y$ is a *certificate* for the membership of $x$ in $L$.

**Example:** If $L$ is SAT, then for a satisfiable expression $x$, a certificate would be a satisfying truth assignment.





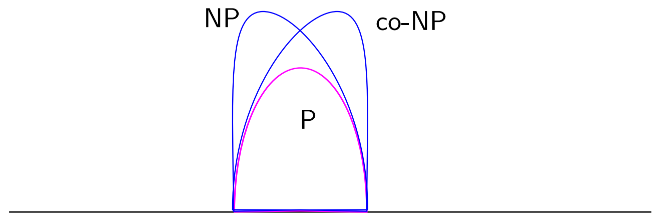
## co-NP

As co-NP is the collection of complements of languages in NP, and P is closed under complementation, co-NP can also be characterised as the collection of languages of the form:

$$L = \{x \mid \forall y\, |y| < p(|x|) \rightarrow R'(x, y)\}$$

NP – the collection of languages with succinct certificates of membership.

co-NP – the collection of languages with succinct certificates of disqualification.




Any of the situations is consistent with our present state of knowledge:

- P = NP = co-NP
- P = NP $\cap$ co-NP $\neq$ NP $\neq$ co-NP
- P $\neq$ NP $\cap$ co-NP = NP = co-NP
- P $\neq$ NP $\cap$ co-NP $\neq$ NP $\neq$ co-NP





## co-NP-complete

VAL – the collection of Boolean expressions that are *valid* is *co-NP-complete*.

Any language $L$ that is the complement of an NP-complete language is *co-NP-complete*.

Any reduction of a language $L_1$ to $L_2$ is also a reduction of $\bar{L_1}$–the complement of $L_1$–to $\bar{L_2}$–the complement of $L_2$.

There is an easy reduction from the complement of SAT to VAL, namely the map that takes an expression to its negation.

$$\text{VAL} \in \text{P} \Rightarrow \text{P} = \text{NP} = \text{co-NP}$$

$$\text{VAL} \in \text{NP} \Rightarrow \text{NP} = \text{co-NP}$$





## Prime Numbers

Consider the decision problem PRIME:

Given a number $x$, is it prime?

This problem is in co-NP.

$$\forall y(y < x \rightarrow (y = 1 \lor \neg(\text{div}(y, x))))$$

Note again, the algorithm that checks for all numbers up to $\sqrt{n}$ whether any of them divides $n$, is not polynomial, as $\sqrt{n}$ is not polynomial in the size of the input string, which is $\log n$.





## Primality

Another way of putting this is that Composite is in NP.

Pratt (1976) showed that PRIME is in NP, by exhibiting succinct certificates of primality based on:

A number $p > 2$ is *prime* if, and only if, there is a number $r$, $1 < r < p$, such that $r^{p-1} = 1 \bmod p$ and $r^{\frac{p-1}{q}} \neq 1 \bmod p$ for all *prime divisors* $q$ of $p - 1$.