Computation with Real Numbers

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Computing over $\mathbb{F}_2$

$\mathbb{F}_2$	$=$	$\{0, 1\}$ (or $\{\text{TRUE}, \text{FALSE}\}$)
		(with addition modulo 2)
in Turing-machines:		primitive operations
standard type		**bool**
		No Problems!

Computing over $\mathbb{N}$

$\mathbb{N}$	$=$	$\{(0), 1, 2, 3, 4, \ldots\}$
in Turing-machines:		standard encoding either unary or binary
standard type		**int**
Potentially overflow-issues		but these can be circumvented

Computing over $\mathbb{Q}$

$\mathbb{Q}$	$=$	$\{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{2}{3} \ldots\}$
in Turing-machines:		standard encoding
standard type		**int** $\times$ **int** no problems once **int** is fixed

What are real numbers?

Definition

The real numbers $\mathbb{R}$ are the metric closure of the rational numbers $\mathbb{Q}$, i.e. everything that may occur as a limit of a Cauchy sequence of rationals.

Think infinite decimal expansions ($\pi = 3.14159265\ldots$) (for now).
Wait a moment.. shouldn't computations be finite?

A detour: Turing machine vs finite automaton

Are computers models of Turing machines or of finite automata?

Claim: A computer has potentially infinite memory.

Infinite Objects in CS

Programming	Lazy Lists
Theory	Oracles (for Turing machines)

Computability on infinite sequences

Definition

A function $F :\subseteq \sum^{\mathbb{N}} \to \sum^{\mathbb{N}}$ is computable, if any finite prefix of $F(p)$ can uniformly be computed from some finite prefix of sufficient length of p.

So we are done, right?

Definition (Suggestion)

A function $f :\subseteq \mathbb{R} \to \mathbb{R}$ is computable, if there is a computable function $F :\subseteq \sum^{\mathbb{N}} \to \sum^{\mathbb{N}}$ such that $F(p)$ is a decimal expansion of $f(x)$ whenever p is a decimal expansion of x.

Not yet!

Definition (Suggestion)

A function $f :\subseteq \mathbb{R} \to \mathbb{R}$ is computable, if there is a computable function $F ::\subseteq \sum^{\mathbb{N}} \to \sum^{\mathbb{N}}$ such that $F(p)$ is a decimal expansion of $f(x)$ whenever p is a decimal expansion of x.

would imply

Proposition

The function $x \mapsto 3x$ is not computable. (but $x \mapsto 2x$ is!)

Standard representation

Definition

A sequence $(q_i)_{i \in \mathbb{N}} \in \mathbb{Q}^{\mathbb{N}}$ *represents* $x \in \mathbb{R}$, if $|x - q_i| < 2^{-i}$ for all $i \in \mathbb{N}$.
In symbols: $\rho(q) = x$

Definition (final)

A function $f :\subseteq \mathbb{R} \to \mathbb{R}$ is computable, if there is a computable function F such that $\rho(F(p)) = f(\rho(p))$.

Computable functions

The following functions are computable:

1. addition
2. multiplication
3. division
4. sin, cos, tan, ...
5. exp, log
6. ...

Continuous functions

Definition

A function $f :\subseteq \mathbb{R} \to \mathbb{R}$ is continuous, if:

$$\forall x \in \mathrm{dom}(f)\ \forall \varepsilon > 0\ \exists \delta > 0\ \forall y \in \mathrm{dom}(f)$$

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$$

Proposition

Every computable function $f :\subseteq \mathbb{R} \to \mathbb{R}$ *is continuous.*

Continuous functions II

Corollary

The function $=_0 : \mathbb{R} \to \mathbb{R}$ *defined via*

$$=_0 (x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

is not computable.

More general: Decision is impossible for real numbers.

Higher types

We can represent the set $\mathcal{C}(\mathbb{R}, \mathbb{R})$ of continuous functions on real numbers by infinite sequences, too! (e.g. by a fast converging sequence of polynomials or polygons with rational coefficients)

$$\Rightarrow$$

We can compute with such functions!

Example

Theorem ((monotone) Intermediate Value Theorem)

A (strictly monotone) continuous function $f : [0,1] \to \mathbb{R}$ *with* $f(0) < 0 < f(1)$ *has a zero, i.e.* $\exists x \in [0,1]$ *such that* $f(x) = 0$*.*

Definition

Consider the problem IVT (m-IVT) of computing a zero given a function satisfying the conditions of the (monotone) Intermediate Value Theorem.

m-IVT: Textbook algorithm

Bisection

1. Compute $f(0.5)$.
2. If $f(0.5) = 0$, then 0.5 is the solution.
3. If $f(0.5) < 0$, then recursively work on the interval $[0.5, 1]$.
4. If $f(0.5) > 0$, then recursively work on the interval $[0, 0.5]$.

Not computable!

m-IVT: Algorithm that works

Trisection

1. Compute $f(0.3)$ and $f(0.7)$.
2. Simultaneously search for a bound away from 0.
3. If proof of $f(0.3) < 0$ has been found, recursively work on the interval $[0.3, 1]$.
4. If proof of $f(0.3) > 0$ has been found, recursively work on the interval $[0, 0.3]$.
5. If proof of $f(0.7) < 0$ has been found, recursively work on the interval $[0.7, 1]$.
6. If proof of $f(0.7) > 0$ has been found, recursively work on the interval $[0, 0.7]$.

Results

Theorem
m-IVT is computable.

Theorem
IVT is not computable.

Philosophical Differences

Polish school
uncomputable points exist
reals are represented by infinite sequences
effective analysis compatible with classical analysis

vs

Russian school
uncomputable points do not exist
reals are represented by finite sequences
effective analysis incompatible with classical analysis

The Textbook

K. Weihrauch
Computable Analysis.
Springer, 2000.