Interactive Formal Verification
6: Sets

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Set Notation in Isabelle

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- Set-theoretic abstractions naturally express many complex constructions.
- A set in higher-order logic is a *boolean-valued map*.
- Its elements must all have the *same type*

Set Theory Primitives

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- The subset relation: $\subseteq$
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- The empty set: `{ }`
- The universal set: `UNIV`

Basic Set Theory Operations

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\begin{aligned}
e \in \{x.\, P(x)\} &\iff P(e) \\
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\begin{aligned}
e \in \{x.\, P(x)\} &\iff P(e) \\
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e \in \texttt{Pow}(A) &\iff e \subseteq A
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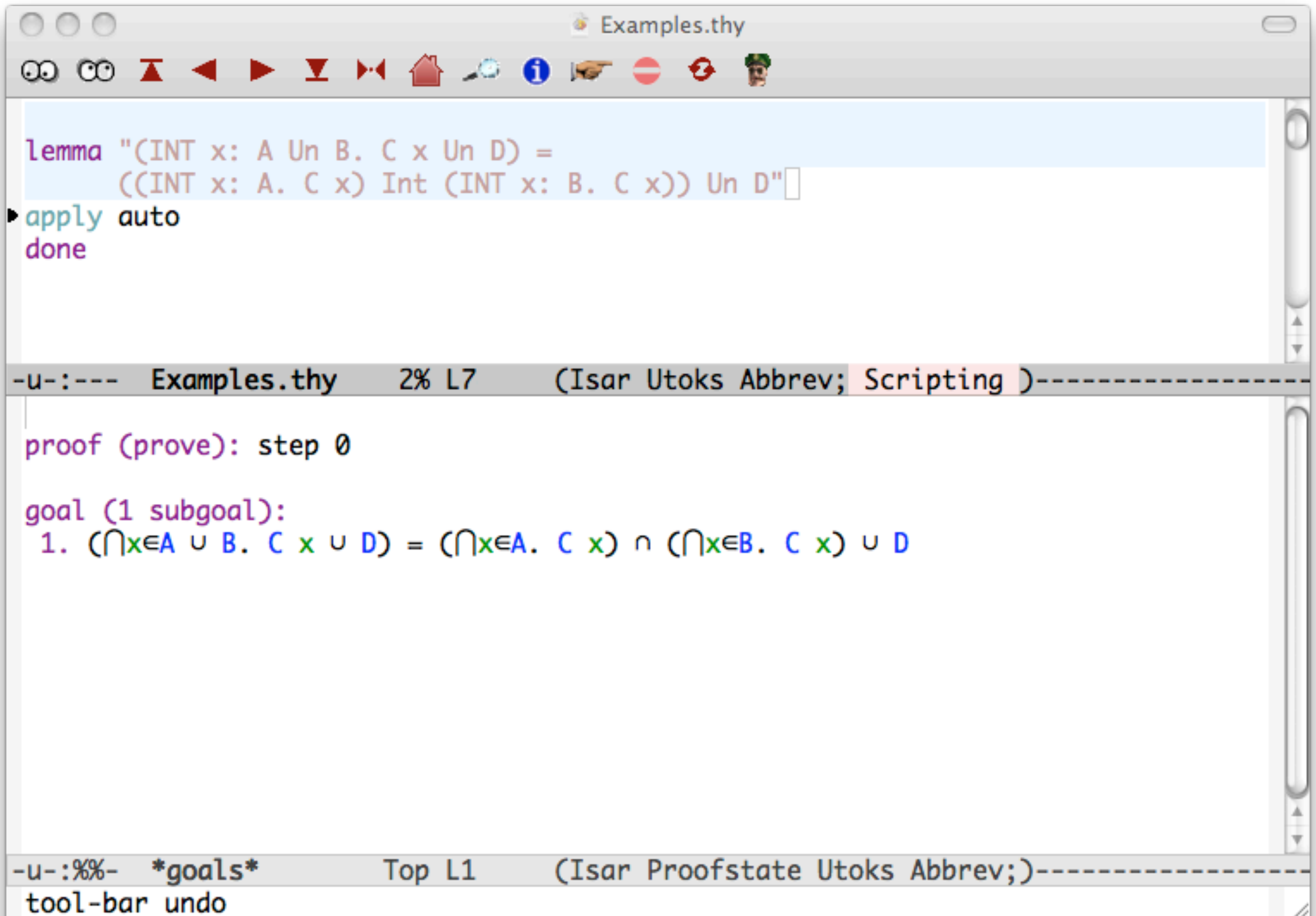
And the analogous forms of intersections...

A Simple Set Theory Proof



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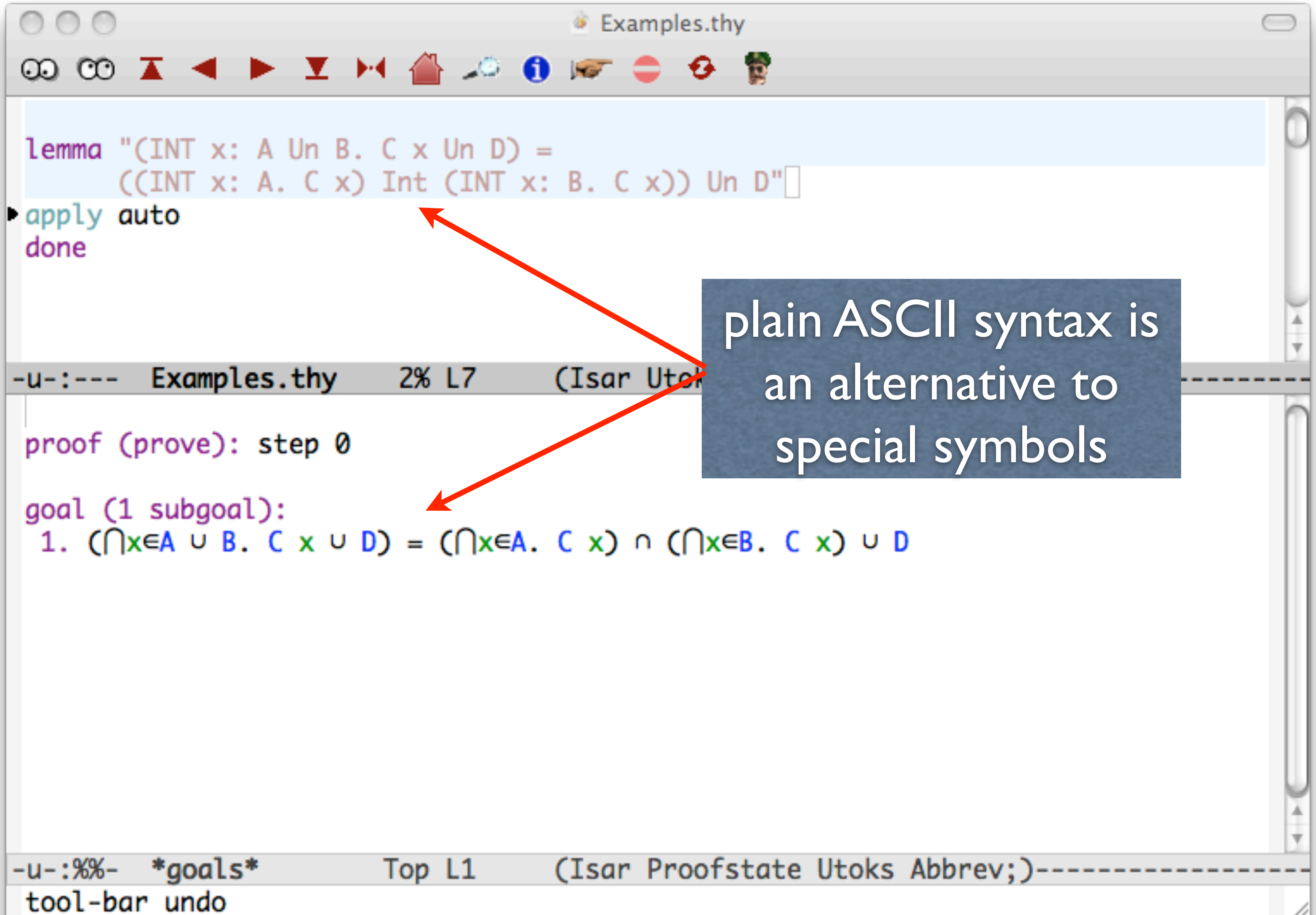

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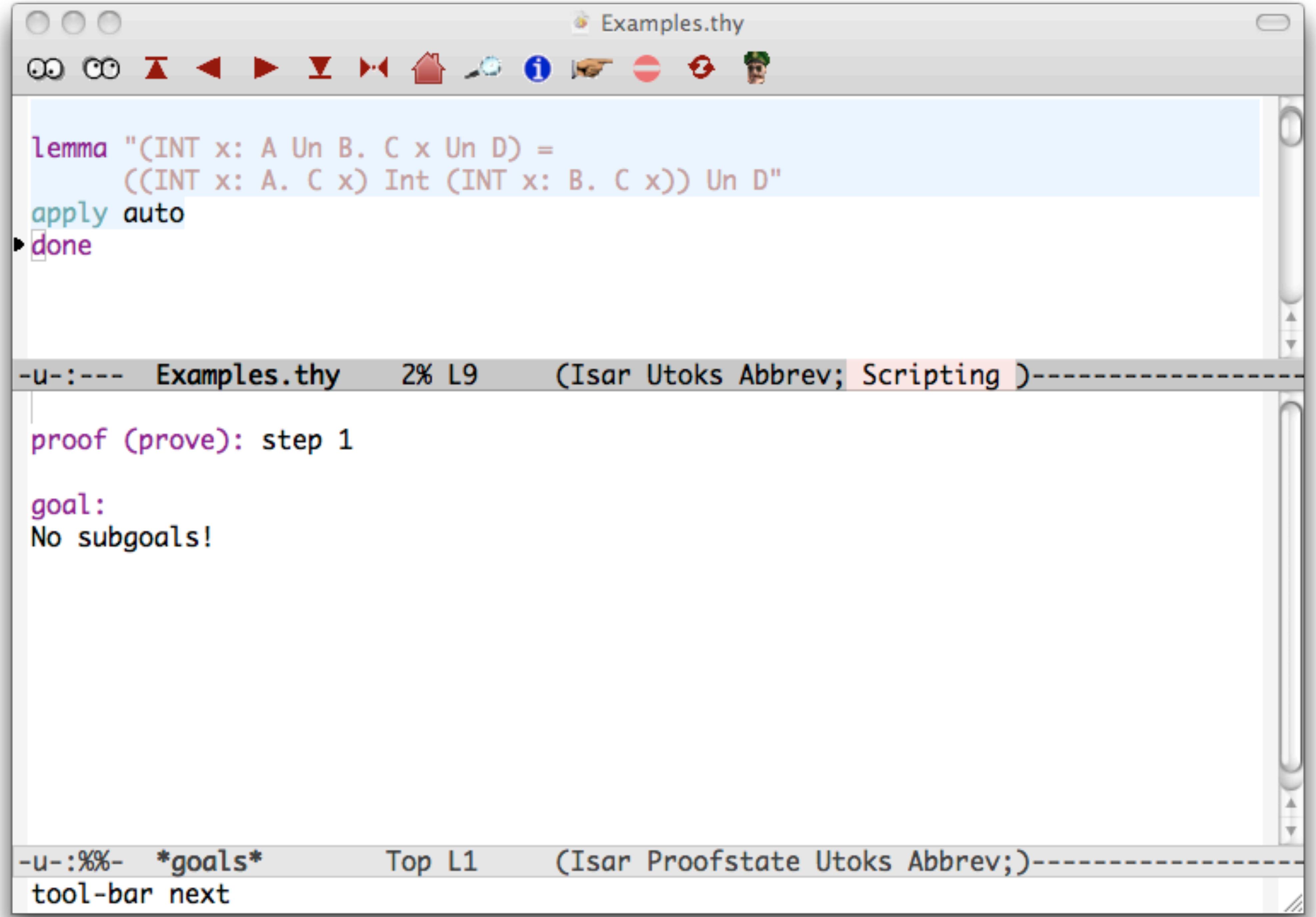

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- Don't *re-invent* image and inverse image!!

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A finite set is defined *inductively*
in terms of `{}` and `insert`

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$$\texttt{finite}(A \cup B) = (\texttt{finite}\, A \wedge \texttt{finite}\, B)$$

$$\texttt{finite}\, A \Longrightarrow \texttt{card}(\texttt{Pow}\, A) = 2^{\texttt{card}\, A}$$

Intervals, Sums and Products

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```
{..<u}  == {x. x < u}
 {..u}  == {x. x ≤ u}
{l<..}  == {x. l<x}
 {l..}  == {x. l≤x}
{l<..<u} == {l<..} ∩ {..<u}
 {l..<u} == {l..} ∩ {..<u}
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`setsum f A` and `setprod f A`

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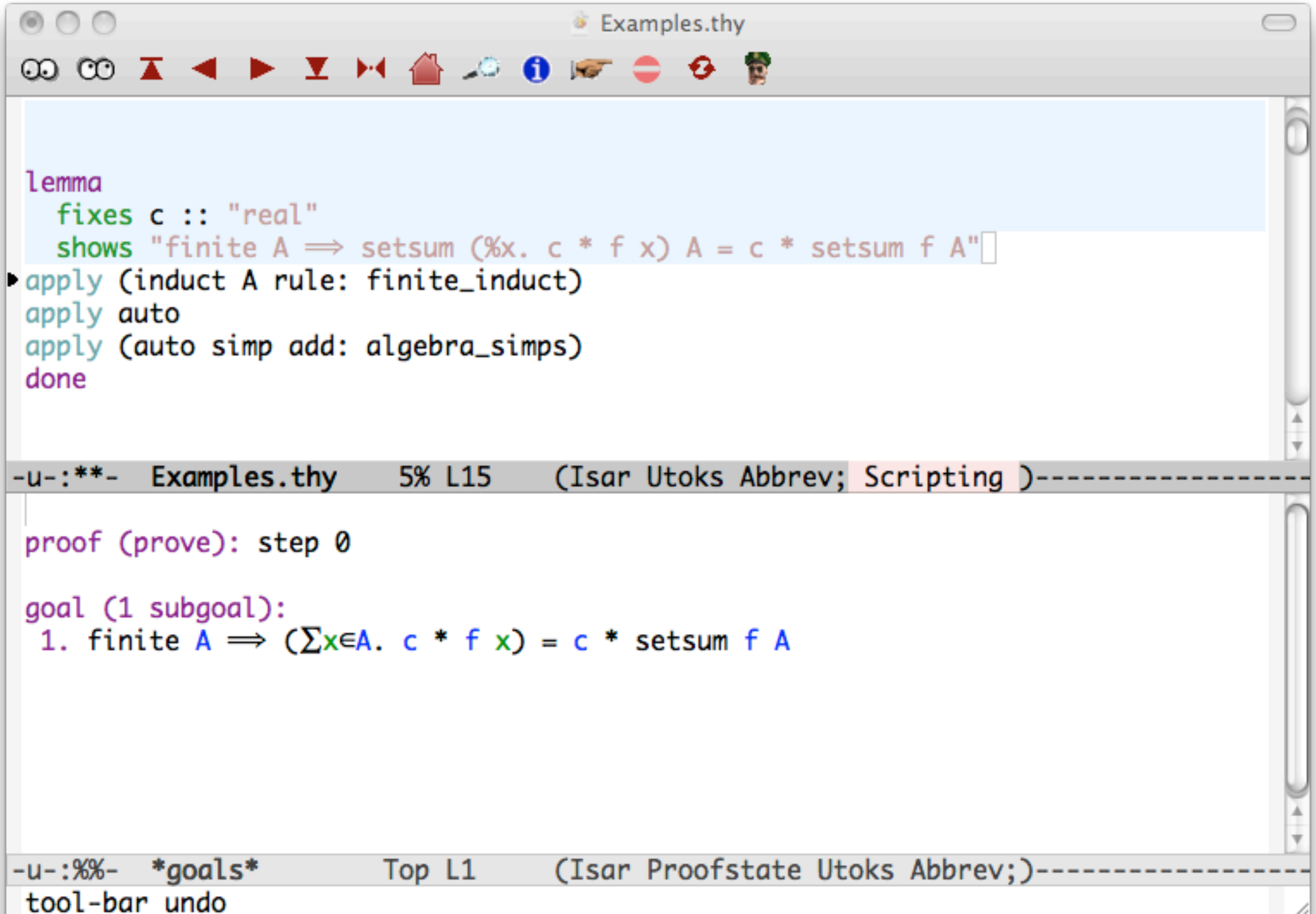
`setsum f A` and `setprod f A`

$\sum i \in I.\ f$ and $\prod i \in I.\ f$

A Harder Proof Involving Sets



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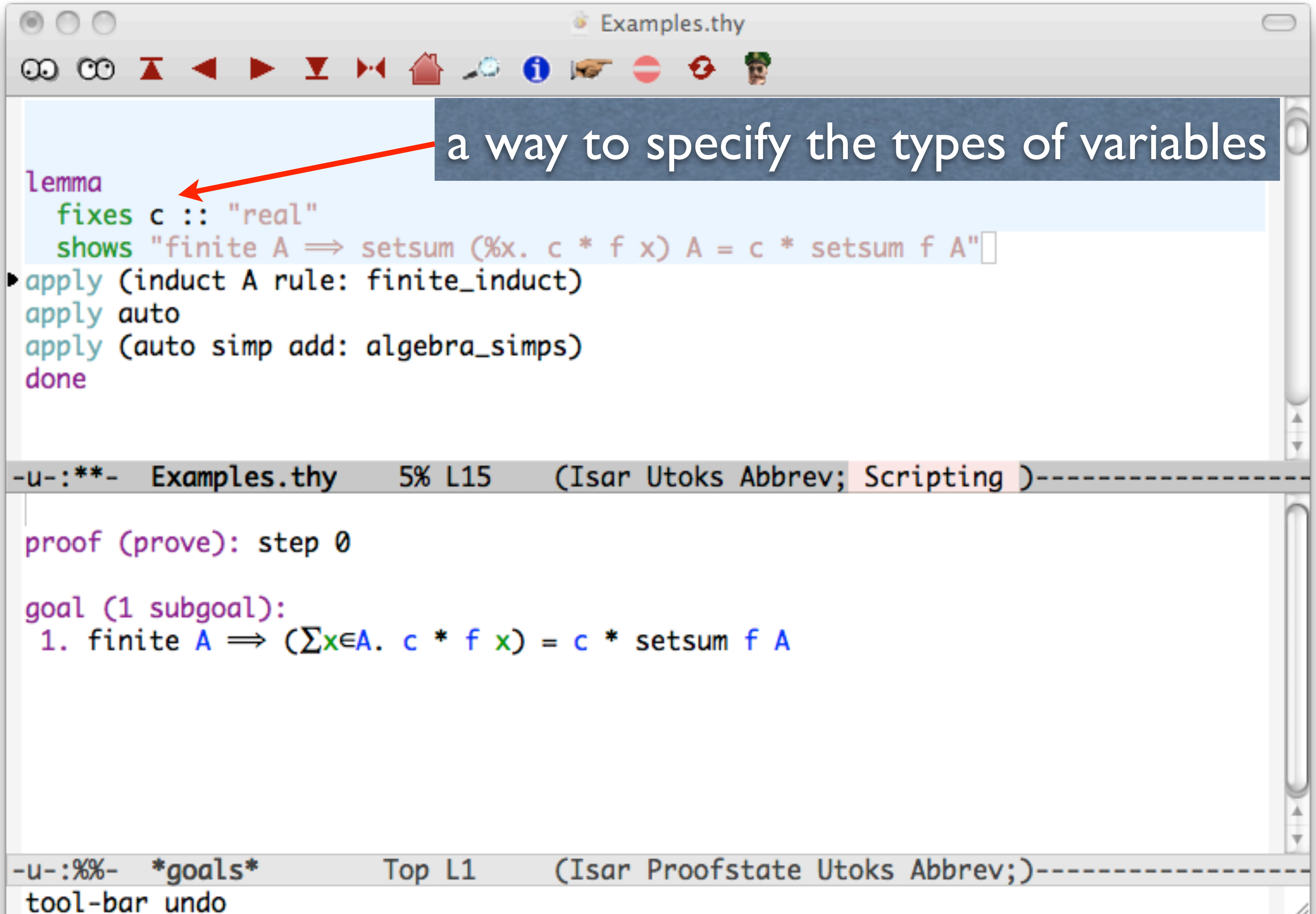

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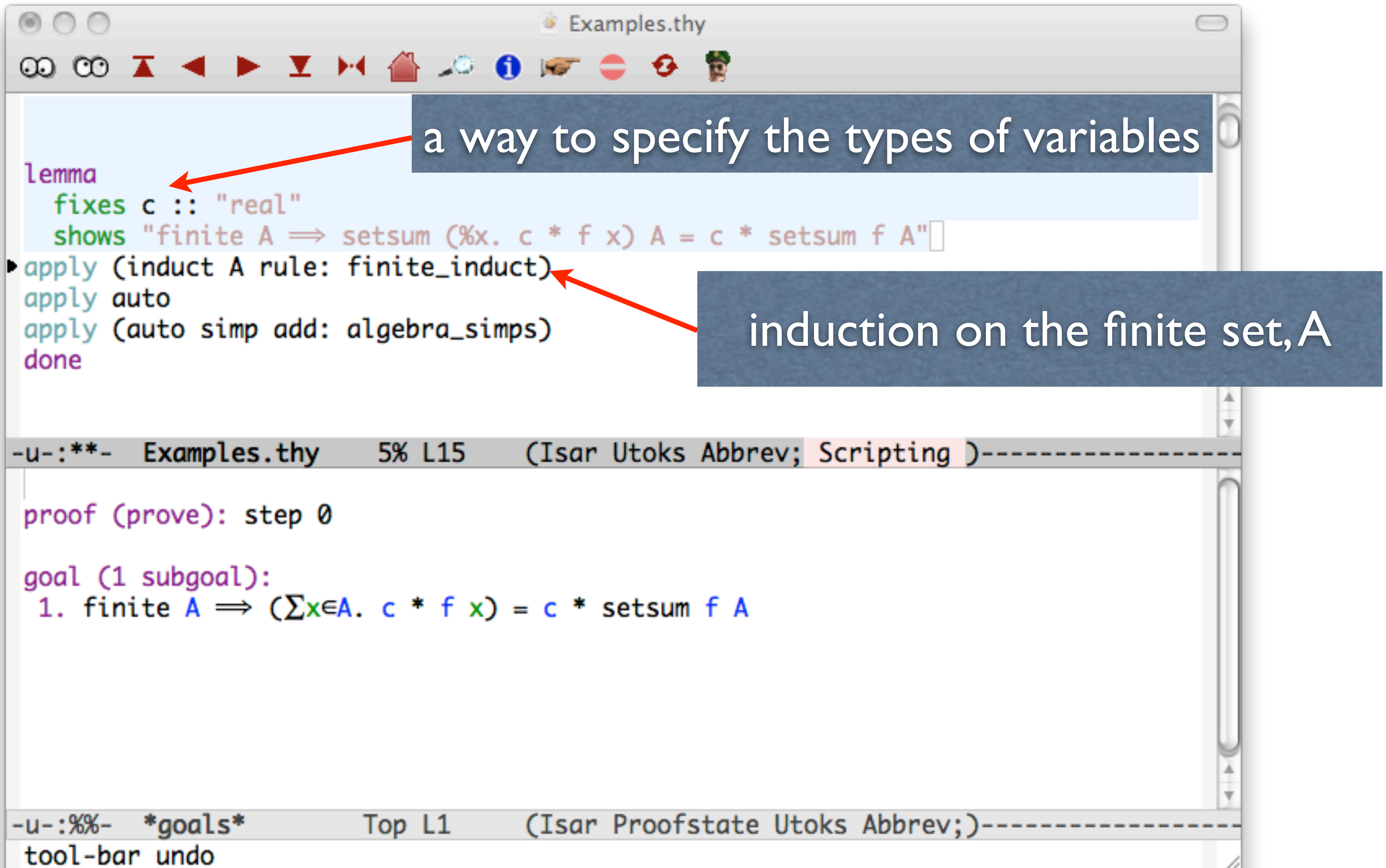


Outcome of the Induction

```
Examples.thy

lemma
  fixes c :: "real"
  shows "finite A ⟹ setsum (%x. c * f x) A = c * setsum f A"
apply (induct A rule: finite_induct)
apply auto
apply (auto simp add: algebra_simps)
done

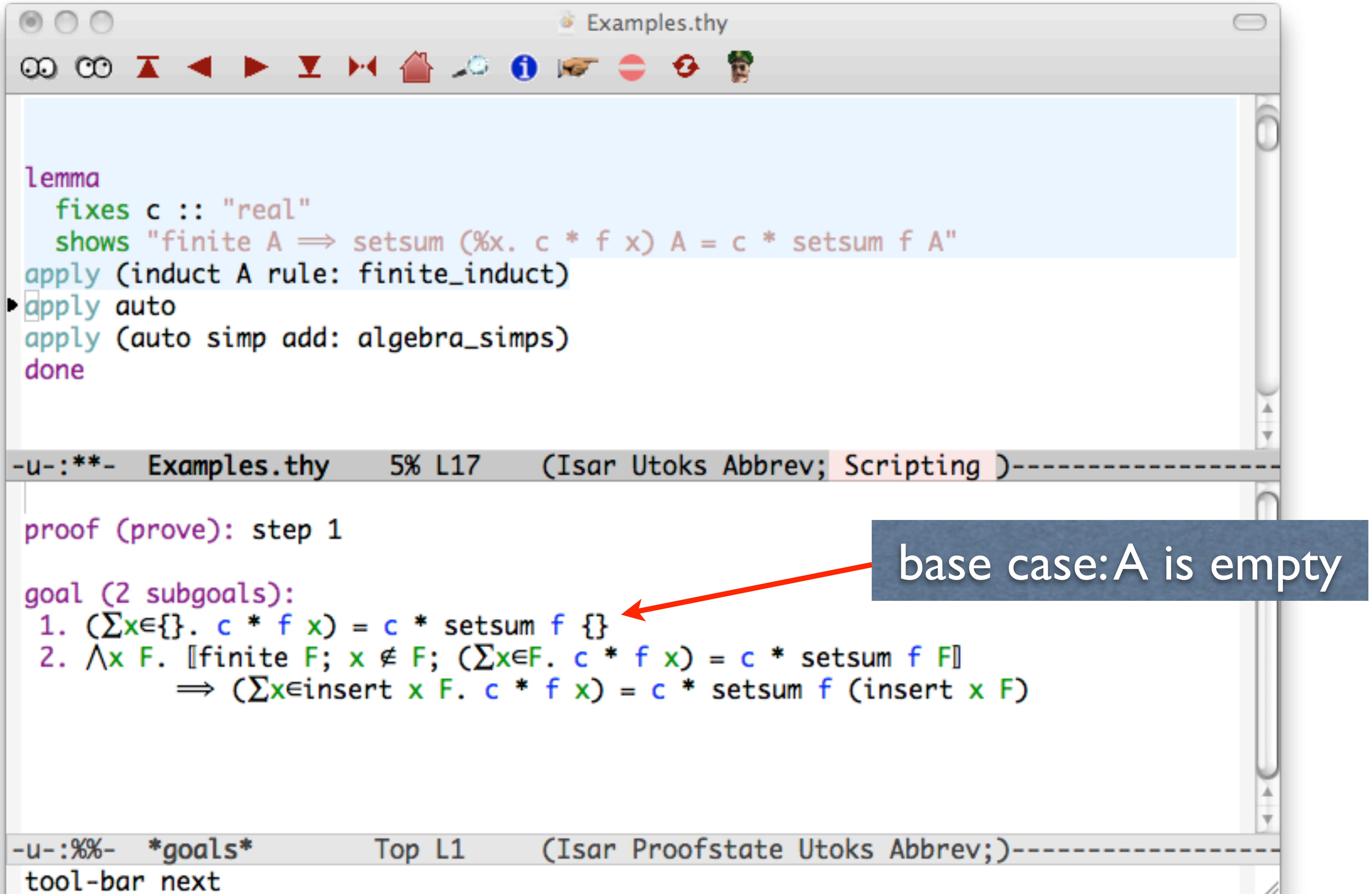
-u-:**-  Examples.thy     5% L17     (Isar Utoks Abbrev; Scripting )----------------

proof (prove): step 1

goal (2 subgoals):
 1. (∑x∈{}. c * f x) = c * setsum f {}
 2. ⋀x F. ⟦finite F; x ∉ F; (∑x∈F. c * f x) = c * setsum f F⟧
          ⟹ (∑x∈insert x F. c * f x) = c * setsum f (insert x F)

-u-:%%-  *goals*        Top L1     (Isar Proofstate Utoks Abbrev;)----------------
tool-bar next
```

Outcome of the Induction



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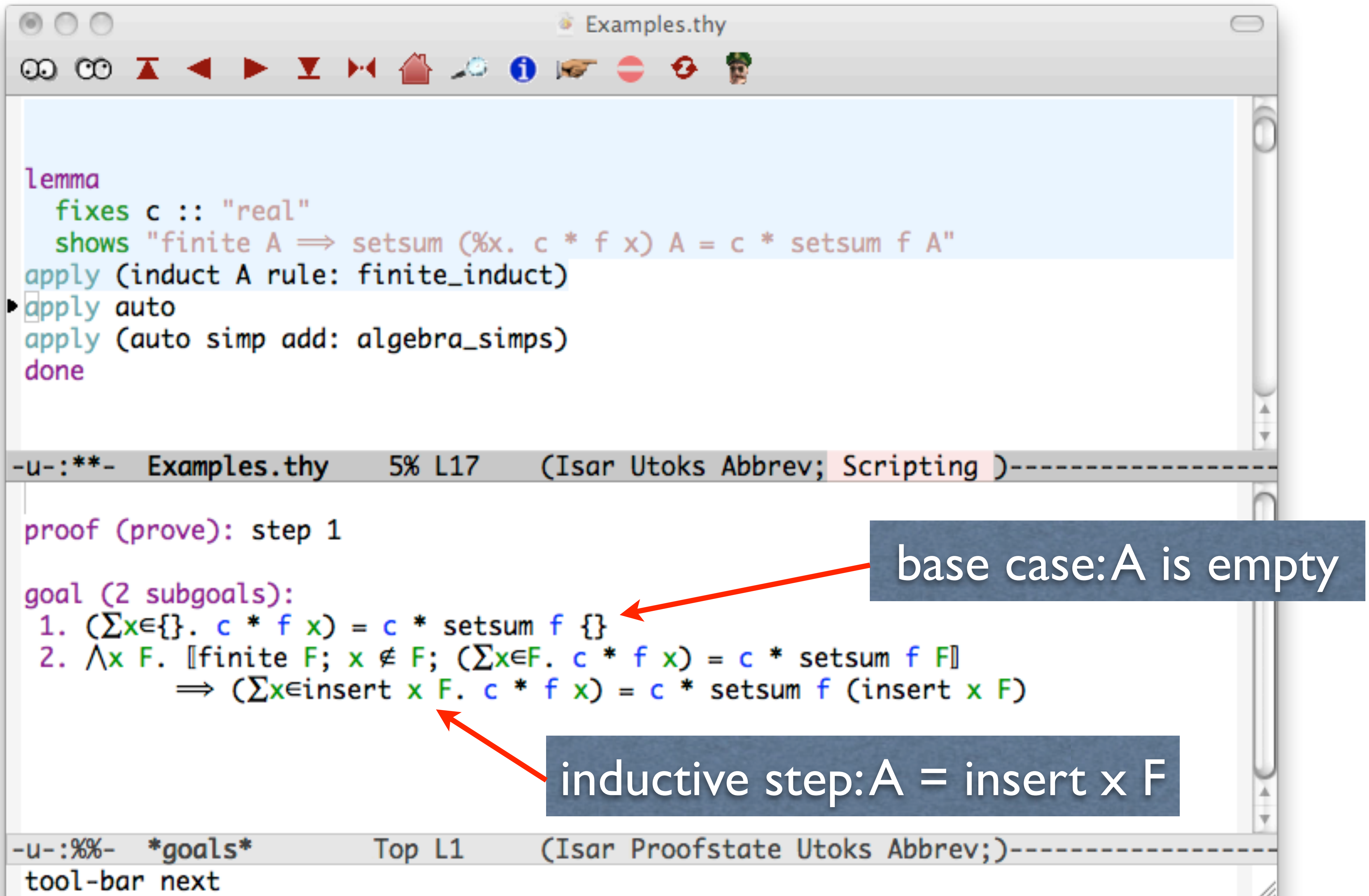
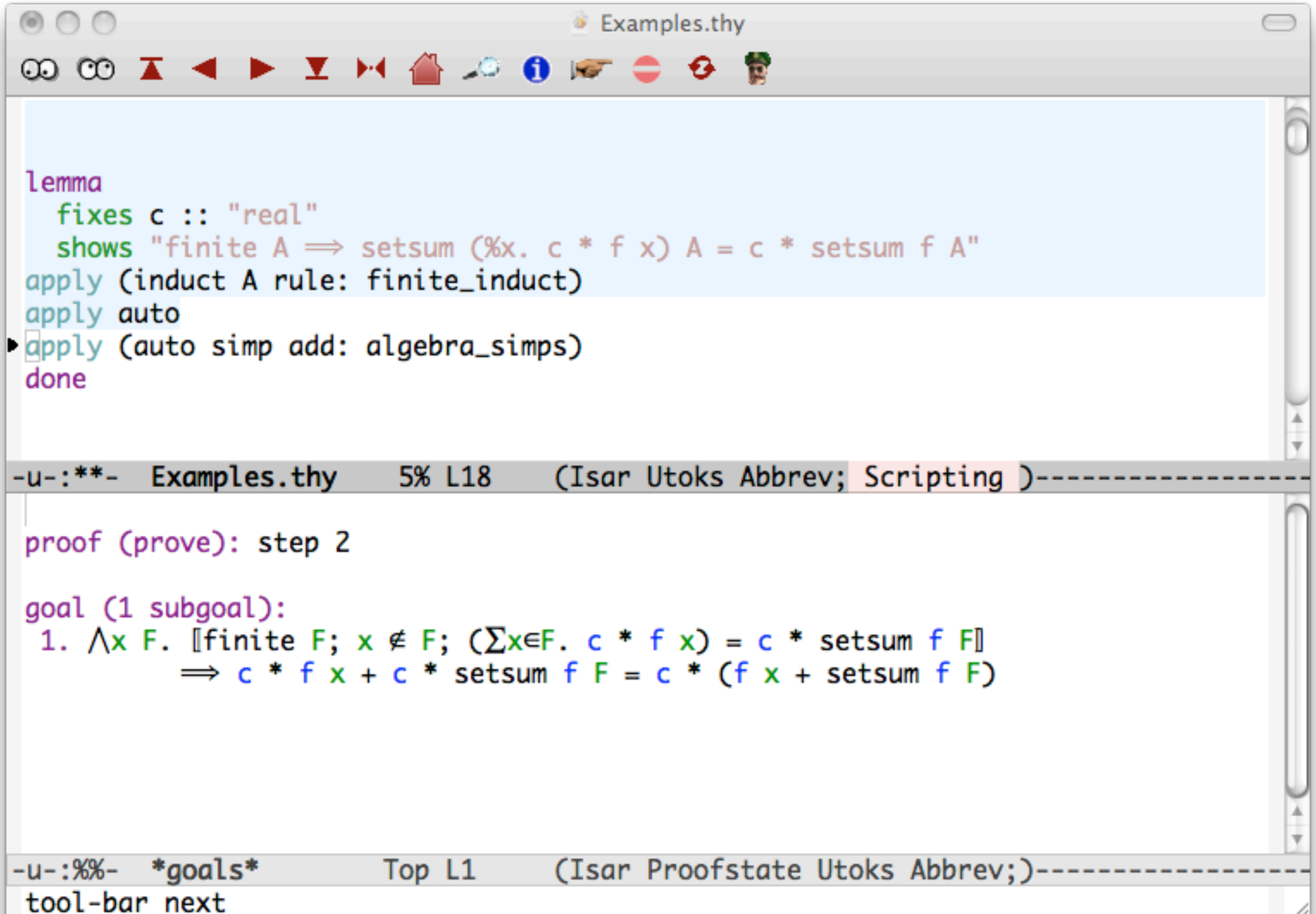


Almost There!



Almost There!

Examples.thy

```
lemma
  fixes c :: "real"
  shows "finite A ⟹ setsum (%x. c * f x) A = c * setsum f A"
apply (induct A rule: finite_induct)
apply auto
apply (auto simp add: algebra_simps)
done
```

```
-u-:**-  Examples.thy     5% L18     (Isar Utoks Abbrev; Scripting )-----------------
```

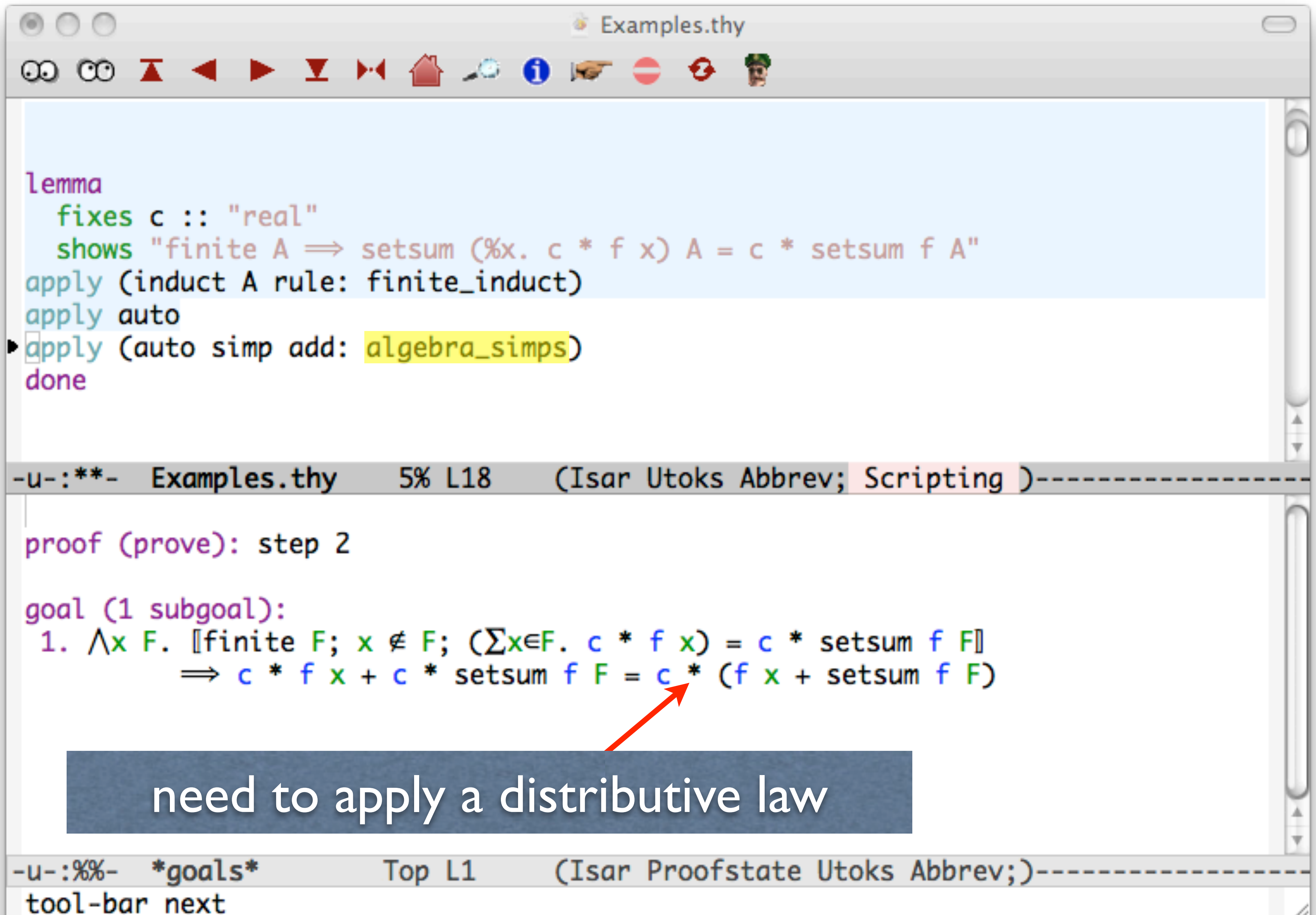
```
proof (prove): step 2

goal (1 subgoal):
 1. ⋀x F. ⟦finite F; x ∉ F; (∑x∈F. c * f x) = c * setsum f F⟧
         ⟹ c * f x + c * setsum f F = c * (f x + setsum f F)
```

need to apply a distributive law

```
-u-:%%-  *goals*        Top L1     (Isar Proofstate Utoks Abbrev;)-----------------
tool-bar next
```

Almost There!



Finished!

```
Examples.thy

lemma
  fixes c :: "real"
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apply auto
apply (auto simp add: algebra_simps)
done

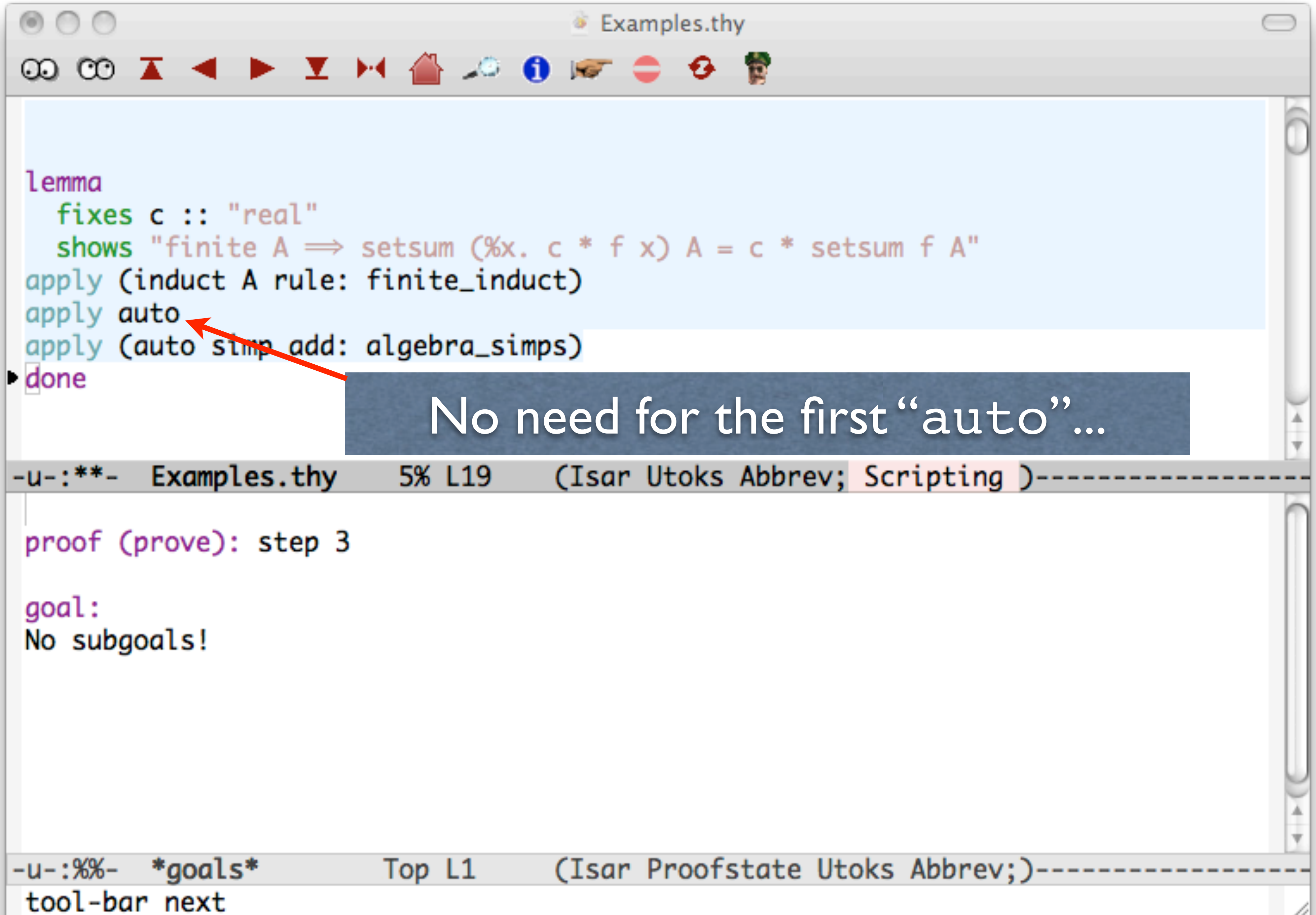
-u-:**-  Examples.thy    5% L19    (Isar Utoks Abbrev; Scripting )----------------

proof (prove): step 3

goal:
No subgoals!

-u-:%%-  *goals*        Top L1    (Isar Proofstate Utoks Abbrev;)----------------
tool-bar next
```

Finished!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lemma
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shows "finite A ⟹ setsum (%x. c * f x) A = c * setsum f A"
apply (induct A rule: finite_induct)
apply auto
apply (auto simp add: algebra_simps)
done
No need for the first "auto"...
-u-:**- Examples.thy 5% L19 (Isar Utoks Abbrev; Scripting)
proof (prove): step 3
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No subgoals!
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Proving Theorems about Sets

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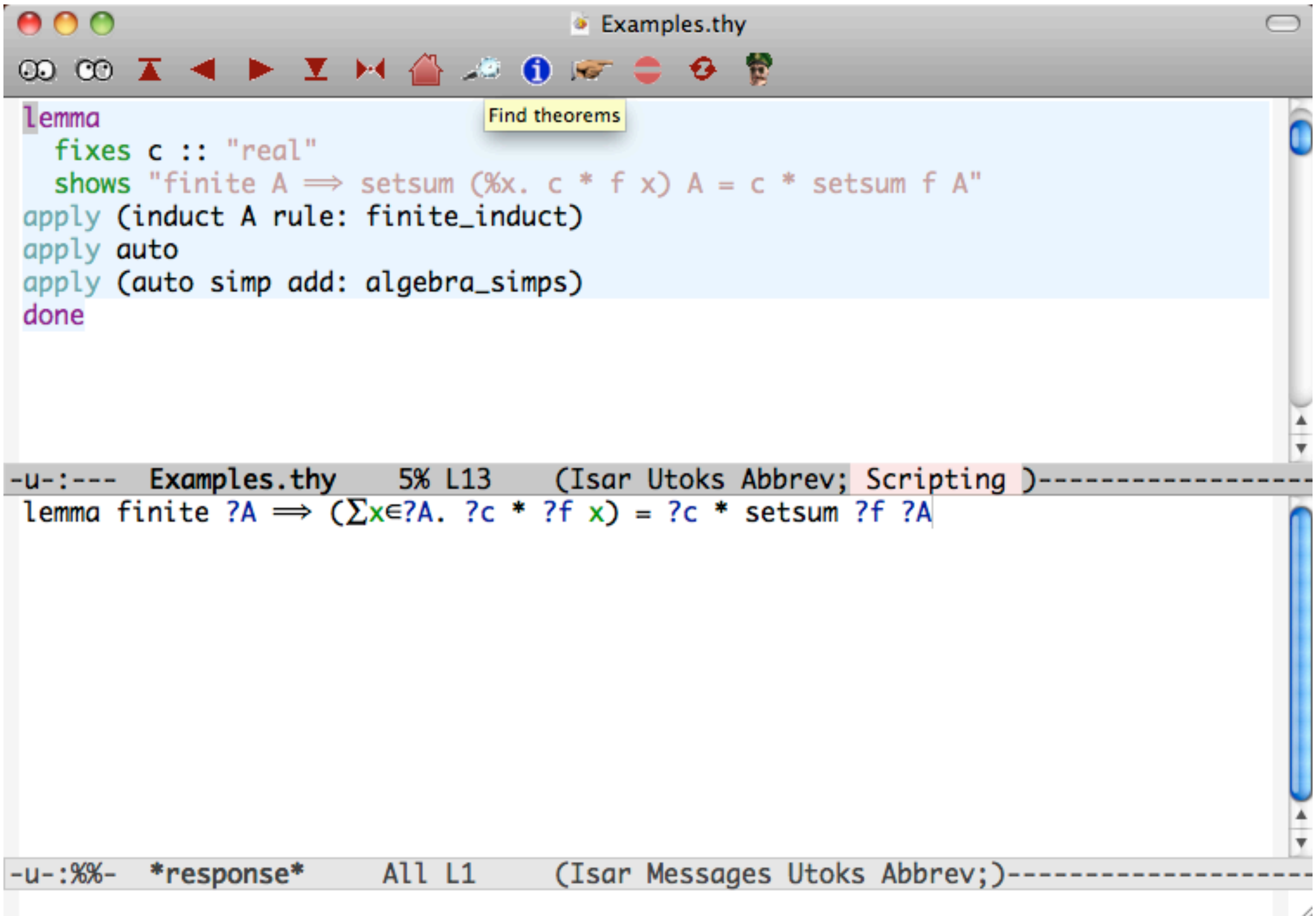
- It is not practical to learn all the built-in lemmas.
- Instead, try an automatic proof method:
  - `auto`
  - `force`
  - `blast`
- Each uses the built-in library, comprising hundreds of facts, with powerful heuristics.

Finding Theorems about Sets



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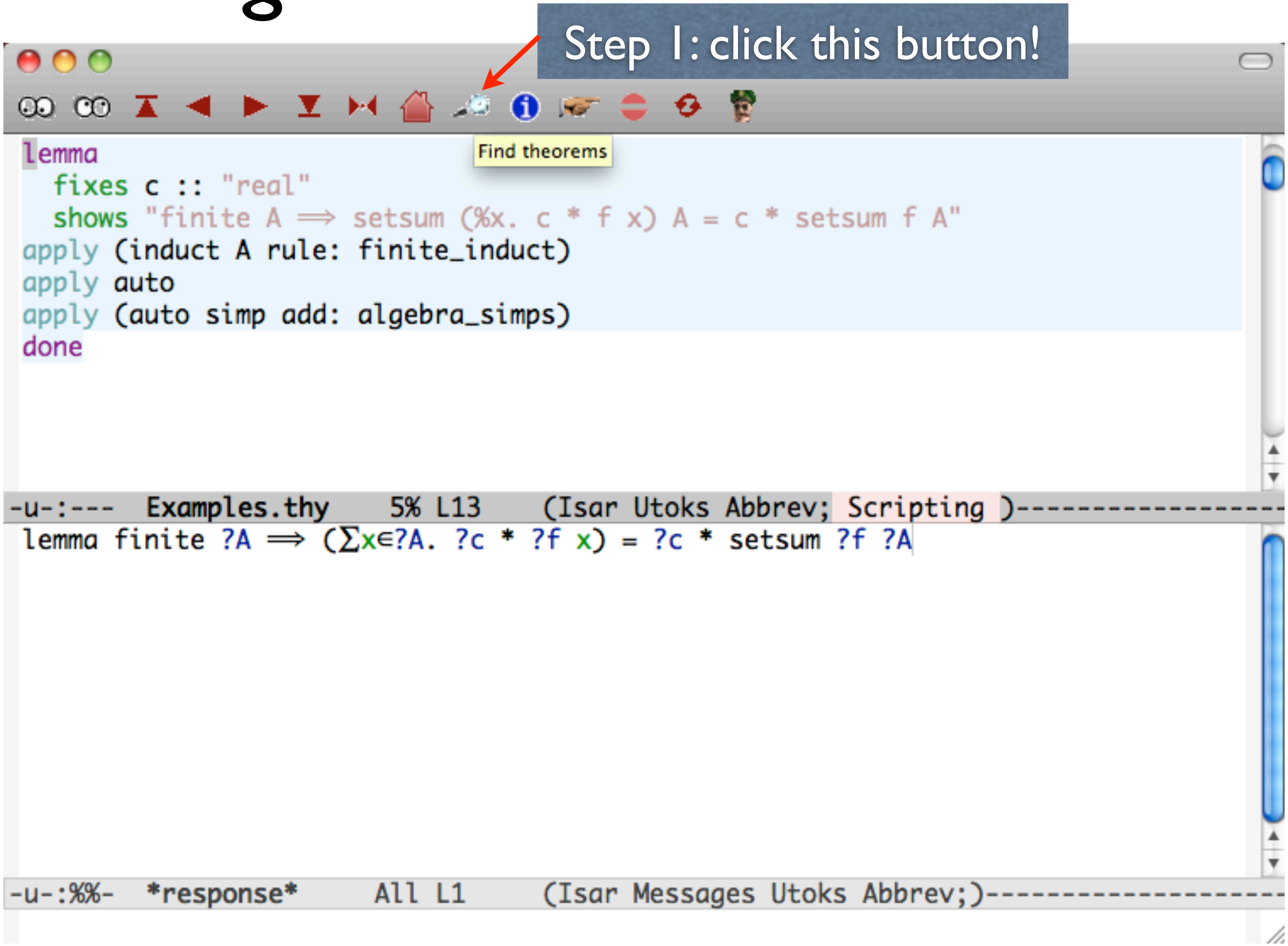
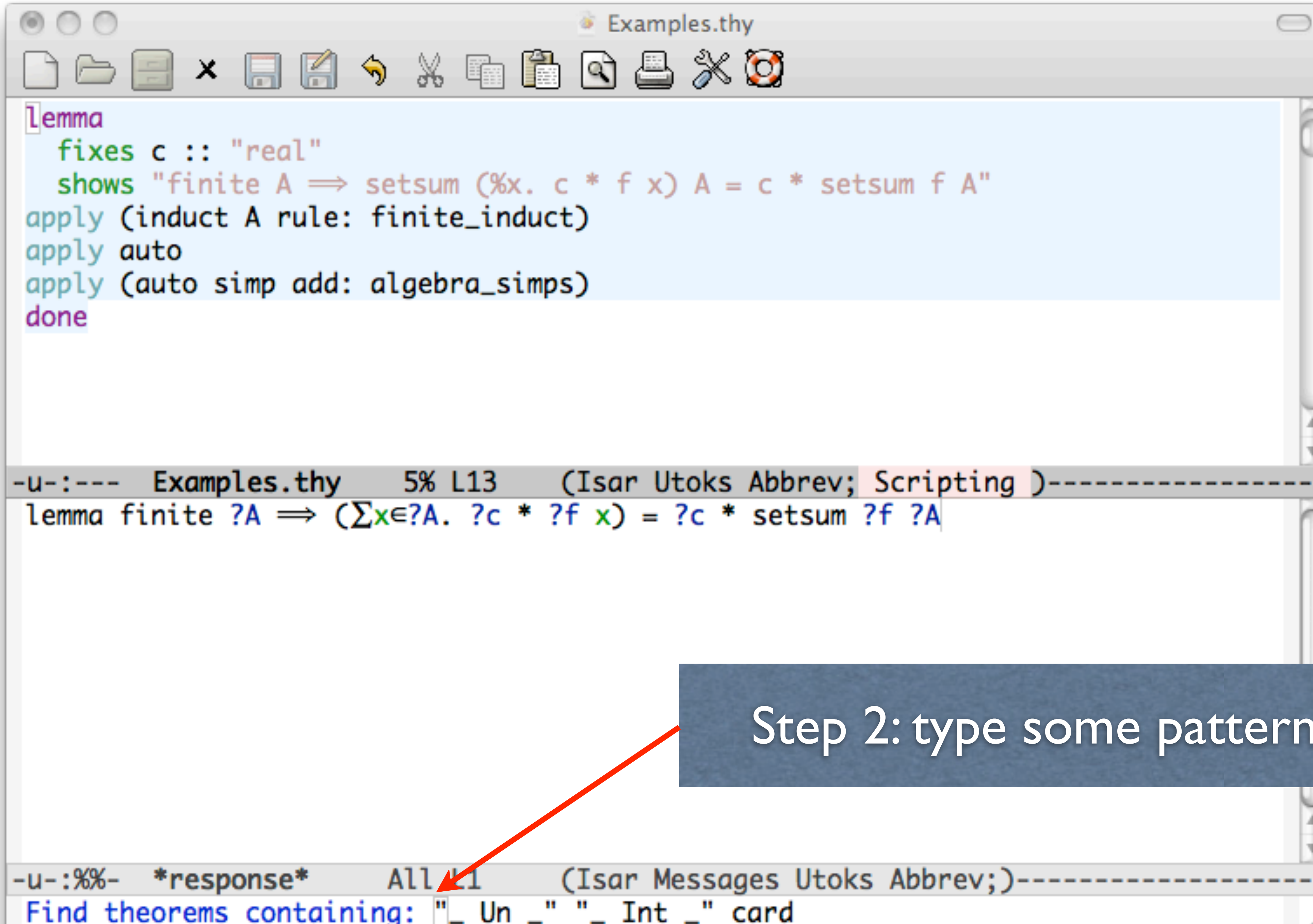


Finding Theorems about Sets



Step 2: type some patterns

What Theorems Were Found?

```
*response*

searched for:
    "_ ∪ _"
    "_ ∩ _"
    "card"

found 2 theorems in 0.120 secs:

Finite_Set.card_Un_Int:
  ⟦finite ?A; finite ?B⟧
  ⟹ card ?A + card ?B = card (?A ∪ ?B) + card (?A ∩ ?B)
Finite_Set.card_Un_disjoint:
  ⟦finite ?A; finite ?B; ?A ∩ ?B = {}⟧ ⟹ card (?A ∪ ?B) = card ?A + card ?B

-u-:%%-  *response*     All L2     (Isar Messages Utoks Abbrev;)--------------------
```