Categorical Logic
Exercise Sheet 4

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Revised: March 8, 2010, 11:23.

Subobject classifiers.

A monomorphism $m\colon S \rightarrowtail A$ in a category is said to be *regular* if it is an equalizer of a two morphisms $A \rightrightarrows B$ into some object B.

1. Show that in a category with finite limits and a subobject classifier, every monomorphism is regular.

   Hint: consider a monomorphism $m\colon S \rightarrowtail A$ and let $\chi\colon A \to \Omega$ be its characteristic map. Then show that m is an equalizer for the morphisms $\chi\colon A \to \Omega$ and $\mathrm{t}_A\colon A \to \Omega$, where t_A is the composite $A \xrightarrow{!} 1 \xrightarrow{\mathrm{t}} \Omega$.

2. Show that if a regular monomorphism is also an epimorphism, then it is an isomorphism.

3. Use (1) and (2) to show that in a category with finite limits and a subobject classifier, every epimorphism is a cover.

4. Conclude from (1) and (2) that the category of monoids does not have a subobject classifier. *Hint:* consider the monoid homomorphism in Exercise 1 of Sheet 3.

5. Show that the category of posets does not have a subobject classifier. *Hint:* describe a monotone function between the following posets that is both monic and epic, but not an isomorphism.

$$\begin{tikzcd} \bullet & & \bullet \\ \bullet & & \bullet \arrow[u] \end{tikzcd}$$

6. Optional: Show that the category of pointed sets has a subobject classifier. Is it a topos?

   Hints: Finite limits of pointed sets can be described in terms of the finite limits of the underlying sets. A subobject of a pointed set is a subset that contains the point. For a subobject classifier, consider the pointed set with two elements. When thinking about whether you have a topos, consult exercise 3(f) on Sheet 1.