MPhil ACS 2010-11
Category Theory for Computer Science
Mid-Term Test

1. On Galois connections

A Galois connection

$$f \dashv g : \mathsf{Q} \to \mathsf{P}$$

between preorders $\mathsf{P} = (P, \leq_{\mathsf{P}})$ and $\mathsf{Q} = (Q, \leq_{\mathsf{Q}})$ is a pair of functions $f : \mathsf{P} \to \mathsf{Q}$ and $g : \mathsf{Q} \to \mathsf{P}$ such that

$$\forall\, x \in P, y \in Q\, .\ f(x) \leq_{\mathsf{Q}} y \iff x \leq_{\mathsf{P}} g(y)\ . \tag{1}$$

(a) For a Galois connection $f \dashv g : \mathsf{Q} \to \mathsf{P}$, show that both f and g are monotone.

(b) For a set X, let $\mathsf{P}(X) = (\mathcal{P}(X), \subseteq)$ be the poset of subsets of X ordered by inclusion. Let $f : X \to Y$ be a function and write $f^{-1}[\_]$ for the inverse-image function $\mathcal{P}(Y) \to \mathcal{P}(X)$.

(1) Define a function $\exists_f : \mathcal{P}(X) \to \mathcal{P}(Y)$ and prove that

$$\exists_f \dashv f^{-1}[\_] : \mathsf{P}(Y) \to \mathsf{P}(X)$$

is a Galois connection.

(2) Define a function $\forall_f : \mathcal{P}(X) \to \mathcal{P}(Y)$ and prove that

$$f^{-1}[\_] \dashv \forall_f : \mathsf{P}(X) \to \mathsf{P}(Y)$$

is a Galois connection.

2. On distributive categories

Let $\mathcal{S}$ be a cartesian category, and fix an object $S \in \mathcal{S}$.

An S-action (A, α) consists of an object $A \in \mathcal{S}$ and a morphism $\alpha : S \times A \to A$ in $\mathcal{S}$.

Define S-act to be the category with

objects given by S-actions,

morphisms $h : (A, \alpha) \to (B, \beta)$ given by maps $h : A \to B$ in $\mathcal{S}$ such that

$$\begin{tikzcd}
S \times A \arrow[r, "\mathrm{id}_S \times h"] \arrow[d, "\alpha"'] & S \times B \arrow[d, "\beta"] \\
A \arrow[r, "h"'] & B
\end{tikzcd} ,$$

identities and composition given as in $\mathcal{S}$.

Show that:

(a) S-act is cartesian.

(b) S-act is a distributive category whenever so is $\mathcal{S}$.

3. On sections and regular, strong, and extremal monomorphisms

- A monomorphism $m : X \to Y$ is *regular* if there exist $f, g : Y \to Z$ such that

$$\begin{tikzcd} X \arrow[r, "m"] & Y \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Z \end{tikzcd}$$

  is an equaliser.

- A monomorphism $m : X \to Y$ is *strong*, if for every commutative square as on the left below where $e : U \to V$ is an epimorphism

$$\begin{tikzcd} U \arrow[r, "e"] \arrow[d, "u"'] & V \arrow[d, "v"] \\ X \arrow[r, "m"'] & Y \end{tikzcd} \qquad\qquad \begin{tikzcd} U \arrow[r, "e"] \arrow[d, "u"'] & V \arrow[d, "v"] \arrow[dl, dashed, "d"'] \\ X \arrow[r, "m"'] & Y \end{tikzcd}$$

  there exists a unique $d : V \to X$ as on the right above such that both triangles commute.

- A monomorphism $m : X \to Y$ is *extremal* if for every commutative triangle

$$\begin{tikzcd} & V \arrow[d, "v"] \\ X \arrow[ur, "e"] \arrow[r, "m"'] & Y \end{tikzcd}$$

  where $e : X \to V$ is an epimorphism, e is an isomorphism.

(a) Prove that the following implications between properties of monomorphisms hold in any category:

$$\text{section} \Longrightarrow \text{regular} \Longrightarrow \text{strong} \Longrightarrow \text{extremal} \; .$$

[None of the above implications is in general an equivalence, but that is another story.]

4. On retractions, sections, pushouts, and coequalisers

Let $r : A \to B$ be a retraction with section $s : B \to A$, so that $r \circ s = \mathrm{id}_B$.

(a) Show that

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "r"'] & C \arrow[d, "q"] \\ B \arrow[r, "p"'] & P \end{tikzcd} \quad \text{is a pushout}$$

if and only if

$$p = q \circ g \circ s \quad \text{and} \quad \begin{tikzcd} A \arrow[r, shift left, "g"] \arrow[r, shift right, "g \circ s \circ r"'] & C \arrow[r, "q"'] & P \end{tikzcd} \text{ is a coequaliser} .$$