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Advanced Data Flow

MPhil in ACS

Background lecture

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Mathematics: Partial Orders, Lattices

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Ordinary numbers (integers, reals, but not complex numbers) are *totally ordered*. We have (for all x, y, z):

$$\begin{aligned} & & x \leq x \\ x \leq y \wedge y \leq z & \Rightarrow & x \leq z \\ x \leq y \wedge y \leq x & \Rightarrow & x = x \end{aligned}$$

and *also*

$$x \leq y \vee y \leq x$$

If we drop the last condition we get a *partial order*. Some values are just *incomparable* (neither $x \leq y$ nor $y \leq x$ holds)

Mathematics: Partial Orders, Lattices (2)

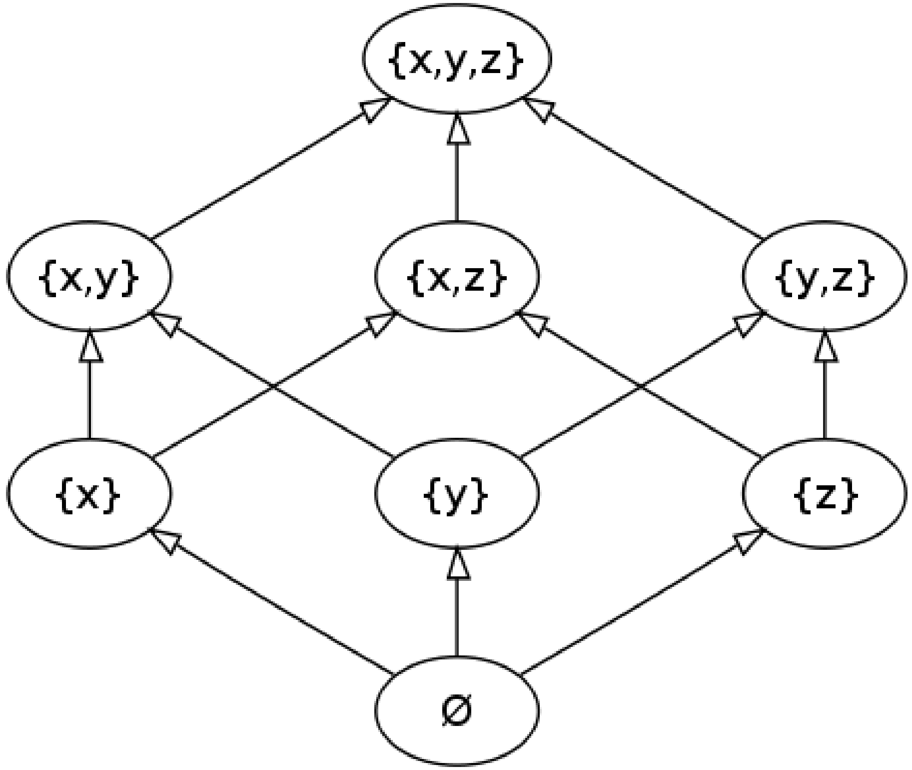
A good example of a partial order are comes from trying to order pairs of numbers (e.g. representing wealth and beauty!). 'Obviously' (x, y) should compare less-than-or-equal to (x', y') provided $x \leq x'$ and $y \leq y'$. But what otherwise?

For cases like $p_1 = (3, 10)$ and $p_2 = (5, 7)$ we say p_1 and p_2 are *incomparable*.

As a notational reminder we usually write $\sqsubseteq$ for partial orders rather than using $\leq$

Another good example of a partial order, and one which is really central to this course is that of ordering sets by subset, e.g. $(\mathcal{P}\{x, y, z\}, \subseteq)$.

Mathematics: Partial Orders, Lattices (3)



Hasse diagram of $(\mathcal{P}\{x, y, z\}, \subseteq)$.

[Image owner: `http://en.wikipedia.org/wiki/User:KSmrq`]

Mathematics: Partial Orders, Lattices (4)

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We all know that $A \cup B = \{x \mid x \in A \vee x \in B\}$

But another way to look at $A \cup B$ is that it is the smallest (w.r.t. $\subseteq$) set C satisfying $A \subseteq C$ and $B \subseteq C$.

In other words it is a *least upper bound* (w.r.t. $\subseteq$) of B and C.

We can use this trick to define analogues of $\cap$ and $\cup$ for other partial orders: for every partial order $(S, \subseteq)$, we can try to define $\sqcup$ (LUB, or 'meet') and $\sqcap$ (GLB, greatest lower bound or 'join') analogously. This doesn't always work, as these may not always exist (find out why), but partial orders for which LUBs and GLBs exist are called *lattices*. $(\mathcal{P}\{x, y, z\}, \subseteq)$ is the prototypical example (especially for this course.)

Lattices

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(Complete) lattices are partial orders $(S, \sqsubseteq)$ for which every subset $X \subseteq S$ has a LUB (written $\bigsqcup X$) and a GLB (written $\sqcap X$). In particular they have a least element $\bot = \sqcap S$ ('bottom') and a greatest element $\top = \bigsqcup S$ ('top').

[(Inessential for this course) technical note: the adjective 'complete' here refers to the fact that X can have an infinite number of elements while LUBs and GLBs on the previous slide used only two. If you've studied an Analysis course in mathematics then note the similarity to limits: considering real numbers ordered by $\leq$ then the LUB of $\{1, 1.9, 1.99, 1.999, \ldots\}$ is 2.]

Monotonicity, fixed points

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An important property of partial orders is *monotonicity*: a function $f : (S, \leq) \rightarrow (S', \leq)$ is monotonic if $x \leq y \Rightarrow f(x) \leq f(y)$.

On reals this is familiar (e.g. the x^3 function is monotonic, but x^2 is not, and neither is $-x$ [think why]).

Given any set S and a function $g : S \rightarrow S$, we say that x is a *fixed point* of g if $g(x) = x$.

A simple example from real numbers: the function $g(x) = ax^2 + (b+1)x + c$ has fixed points $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Tarski-Knaster theorem

[The mathematical basis of the iterative solution of dataflow equations and why this terminates.]

Every monotonic function f on a complete lattice $(X, \sqsubseteq)$ has a least fixed point w.r.t. $\sqsubseteq$ (and a greatest fixed point)

Under all circumstances in this course (technically 'continuity') then this least fixed point can be expressed as $\bigsqcup_{i=0}^{\infty} f^i(\bot)$.

If the lattice has *finite height* (no infinite distinct chains of element $x_0 \sqsubseteq x_1 \sqsubseteq x_2 \sqsubseteq \ldots$) then this limit occurs for a finite value of i.