We know $(a, b) = (b, a) \Rightarrow a = b$ for pairs

so why not lift the result to set product?

Theorem ? $(A \times B = B \times A) \Rightarrow A = B$

Proof?

The first components of the pairs in $A \times B$ are from A.

The first components of the pairs in $B \times A$ are from B.

If $A \times B = B \times A$ then these must be the same, so $A = B$.

slide 129

Theorem ? $(A \times B = B \times A) \Rightarrow A = B$

Proof?

1. Assume $A \times B = B \times A$

We prove $A = B$, i.e. $\forall x.x \in A \Leftrightarrow x \in B$

2. Consider an arbitrary x.

We first prove the $\Rightarrow$ implication.

3. Assume $x \in A$.

4. Consider an arbitrary $y \in B$.

5. $(x, y) \in A \times B$ by defn $\times$

6. $(x, y) \in B \times A$ by 1

7. $x \in B$ by defn $\times$

8. $x \in A \Rightarrow x \in B$ from 3–7 by $\Rightarrow$-introduction

9. The proof of the $\Leftarrow$ implication is symmetric

10. $\forall x.x \in A \Leftrightarrow x \in B$ from 2–9 by $\forall$-introduction

slide 130

Theorem

$(A \times B = B \times A) \wedge (\exists x.x \in A) \wedge (\exists y.y \in B) \Rightarrow A = B$

Proof

1. Assume $A \times B = B \times A \wedge (\exists x.x \in A) \wedge (\exists y.y \in B)$

1a. $A \times B = B \times A$ from 1 by $\wedge$-elimination

1b. $(\exists x.x \in A)$ from 1 by $\wedge$-elimination

1c. $(\exists y.y \in B)$ from 1 by $\wedge$-elimination

We prove $A = B$, i.e. $\forall x.x \in A \Leftrightarrow x \in B$

2. Consider an arbitrary x.

We first prove the $\Rightarrow$ implication.

3. Assume $x \in A$.

4. We have actual $y \in B$ from 1c by $\exists$-elimination

5. $(x, y) \in A \times B$ by defn $\times$

6. $(x, y) \in B \times A$ by 1a

7. $x \in B$ by defn $\times$

8. $x \in A \Rightarrow x \in B$ from 3–7 by $\Rightarrow$-introduction

9. The proof of the $\Leftarrow$ implication is symmetric

10. $\forall x.x \in A \Leftrightarrow x \in B$ from 2–9 by $\forall$-introduction $\square$

slide 131

Theorem

$(A \times B = B \times A) \wedge (\exists x.x \in A) \wedge (\exists y.y \in B) \Rightarrow A = B$

or equivalently

Theorem $(A \times B = B \times A) \Rightarrow A = B \vee A = \emptyset \vee B = \emptyset$

using $((P \wedge R) \Rightarrow Q)$ iff $(P \Rightarrow Q \vee \neg R)$ and De Morgan

slide 132