B Exercises

Exercise Sheet 1: Propositional Logic

1. Let p stand for the proposition "I bought a lottery ticket" and q for "I won the jackpot". Express the following as natural English sentences:
   (a) $\neg p$
   (b) $p \vee q$
   (c) $p \wedge q$
   (d) $p \Rightarrow q$
   (e) $\neg p \Rightarrow \neg q$
   (f) $\neg p \vee (p \wedge q)$
2. Formalise the following in terms of atomic propositions r, b, and w, first making clear how they correspond to the English text.
   (a) Berries are ripe along the path, but rabbits have not been seen in the area.
   (b) Rabbits have not been seen in the area, and walking on the path is safe, but berries are ripe along the path.
   (c) If berries are ripe along the path, then walking is safe if and only if rabbits have not been seen in the area.
   (d) It is not safe to walk along the path, but rabbits have not been seen in the area and the berries along the path are ripe.
   (e) For walking on the path to be safe, it is necessary but not sufficient that berries not be ripe along the path and for rabbits not to have been seen in the area.
   (f) Walking is not safe on the path whenever rabbits have been seen in the area and berries are ripe along the path.
3. Formalise these statements and determine (with truth tables or otherwise) whether they are consistent (i.e. if there are some assumptions on the atomic propositions that make it true): "The system is in a multiuser state if and only if it is operating normally. If the system is operating normally, the kernel is functioning. Either the kernel is not functioning or the system is in interrupt mode. If the system is not in multiuser state, then it is in interrupt mode. The system is not in interrupt mode."
4. When is a propositional formula P *valid*? When is P *satisfiable*?
5. For each of the following propositions, construct a truth table and state whether the proposition is valid or satisfiable. (For brevity, you can just write one truth table with many columns.)
   (a) $p \wedge \neg p$
   (b) $p \vee \neg p$
   (c) $(p \vee \neg q) \Rightarrow q$
   (d) $(p \vee q) \Rightarrow (p \wedge q)$
   (e) $(p \Rightarrow q) \Leftrightarrow (\neg q \Rightarrow \neg p)$
   (f) $(p \Rightarrow q) \Rightarrow (q \Rightarrow p)$
6. For each of the following propositions, construct a truth table and state whether the proposition is valid or satisfiable.

(a) $\mathrm{p} \Rightarrow (\neg \mathrm{q} \vee \mathrm{r})$

(b) $\neg \mathrm{p} \Rightarrow (\mathrm{q} \Rightarrow \mathrm{r})$

(c) $(\mathrm{p} \Rightarrow \mathrm{q}) \vee (\neg \mathrm{p} \Rightarrow \mathrm{r})$

(d) $(\mathrm{p} \Rightarrow \mathrm{q}) \wedge (\neg \mathrm{p} \Rightarrow \mathrm{r})$

(e) $(\mathrm{p} \Leftrightarrow \mathrm{q}) \vee (\neg \mathrm{q} \Leftrightarrow \mathrm{r})$

(f) $(\neg \mathrm{p} \Leftrightarrow \neg \mathrm{q}) \Leftrightarrow (\mathrm{q} \Leftrightarrow \mathrm{r})$

7. Formalise the following and, by writing truth tables for the premises and conclusion, determine whether the arguments are valid.

   (a) Either John isn't stupid and he is lazy, or he's stupid.
   John is stupid.
   Therefore, John isn't lazy.

   (b) The butler and the cook are not both innocent
   Either the butler is lying or the cook is innocent
   Therefore, the butler is either lying or guilty

8. Use truth tables to determine which of the following are equivalent to each other:

   (a) P

   (b) $\neg P$

   (c) $P \Rightarrow F$

   (d) $P \Rightarrow T$

   (e) $F \Rightarrow P$

   (f) $T \Rightarrow P$

   (g) $\neg\neg P$

9. Use truth tables to determine which of the following are equivalent to each other:

   (a) $(P \wedge Q) \vee (\neg P \wedge \neg Q)$

   (b) $\neg P \vee Q$

   (c) $(P \vee \neg Q) \wedge (Q \vee \neg P)$

   (d) $\neg(P \vee Q)$

   (e) $(Q \wedge P) \vee \neg P$

10. Imagine that a logician puts four cards on the table in front of you. Each card has a number on one side and a letter on the other. On the uppermost faces, you can see E, K, 4, and 7. He claims that if a card has a vowel on one side, then it has an even number on the other. How many cards do you have to turn over to check this?

11. Give a truth-table definition of the ternary boolean operation **if** P **then** Q **else** R.

12. Given the truth table for an arbitrary n-ary function $f(\mathrm{p}_1, .., \mathrm{p}_n)$ (from n propositional variables $\mathrm{p}_1, .., \mathrm{p}_n$ to $\{T, F\}$), describe how one can build a proposition, using only $\mathrm{p}_1, .., \mathrm{p}_n$ and the connectives $\wedge$, $\vee$, and $\neg$, that has the same truth table as f. (Hint: first consider each *line* of the truth table separately, and then how to combine them.)

13. Show, by equational reasoning from the axioms in the notes, that $\neg(P \wedge (Q \vee R \vee S))$ iff $\neg P \vee (\neg Q \wedge \neg R \wedge \neg S)$

Exercise Sheet 2: Predicate Logic

1. Formalise the following statements in predicate logic, making clear what your atomic predicate symbols stand for and what the domains of any variables are.
   (a) Anyone who has forgiven at least one person is a saint.
   (b) Nobody in the calculus class is smarter than everybody in the discrete maths class.
   (c) Anyone who has bought a Rolls Royce with cash must have a rich uncle.
   (d) If anyone in the college has the measles, then everyone who has a friend in the college will have to be quarantined.
   (e) Everyone likes Mary, except Mary herself.
   (f) Jane saw a bear, and Roger saw one too.
   (g) Jane saw a bear, and Roger saw it too.
   (h) If anyone can do it, Jones can.
   (i) If Jones can do it, anyone can.
2. Translate the following into idiomatic English.
   (a) $\forall\, x.(\mathrm{H}(x) \wedge \forall\, y.\neg\mathrm{M}(x, y)) \Rightarrow \mathrm{U}(x)$ where $\mathrm{H}(x)$ means x is a man, $\mathrm{M}(x, y)$ means x is married to y, $\mathrm{U}(x)$ means x is unhappy, and x and y range over people.
   (b) $\exists\, z.\mathrm{P}(z, x) \wedge \mathrm{S}(z, y) \wedge \mathrm{W}(y)$ where $\mathrm{P}(z, x)$ means z is a parent of x, $\mathrm{S}(z, y)$ means z and y are siblings, $\mathrm{W}(y)$ means y is a woman, and x, y, and z range over people.
3. State whether the following are true or false, where x, y and z range over the integers.
   (a) $\forall\, x.\exists\, y.(2x - y = 0)$
   (b) $\exists\, y.\forall\, x.(2x - y = 0)$
   (c) $\forall\, x.\exists\, y.(x - 2y = 0)$
   (d) $\forall\, x.x < 10 \Rightarrow \forall\, y.(y < x \Rightarrow y < 9)$
   (e) $\exists\, y.\exists\, z.y + z = 100$
   (f) $\forall\, x.\exists\, y.(y > x \wedge \exists\, z.y + z = 100)$
4. What changes above if x, y and z range over the reals?
5. Formalise the following (over the real numbers):
   (a) Negative numbers don't have square roots
   (b) Every positive number has exactly two square roots

Exercise Sheet 3: Structured Proof

1. Give structured proofs of
   (a) $(P \Rightarrow Q) \Rightarrow ((Q \Rightarrow R) \Rightarrow (P \Rightarrow R))$
   (b) $(P \Rightarrow Q) \Rightarrow ((R \Rightarrow \neg Q) \Rightarrow (P \Rightarrow \neg R))$
   (c) $(P \Rightarrow (Q \Rightarrow R)) \Rightarrow (\neg R \Rightarrow (P \Rightarrow \neg Q))$
2. Consider the following non-Theorem. What's wrong with the claimed proof?

   Non-Theorem Suppose x and y are reals, and $x + y = 10$. Then $x \neq 3$ and $y \neq 8$.

   Proof Suppose the conclusion of the Theorem is false. Then $x = 3$ and $y = 8$. But then $x + y = 11$, which contradicts the assumption that $x + y = 10$. Hence the conclusion must be true.
3. Give a structured proof of $((\forall\, x.\mathrm{L}(x) \Rightarrow \mathrm{F}(x)) \wedge (\exists\, x.\mathrm{L}(x) \wedge \neg \mathrm{C}(x))) \Rightarrow \exists\, x.\mathrm{F}(x) \wedge \neg \mathrm{C}(x)$
4. Give a structured proof of $(\exists\, x.(P(x) \Rightarrow Q(x))) \Rightarrow ((\forall\, x.P(x)) \Rightarrow \exists\, x.Q(x))$
5. Prove that, for any $n \in \mathbb{N}$, n is even iff n^3 is even (hint: first define what 'even' means).
6. Prove that the following are equivalent:
   (a) $\exists\, x.P(x) \wedge \forall\, y.(P(y) \Rightarrow y = x)$
   (b) $\exists\, x.\forall\, y.P(y) \Leftrightarrow y = x$

Exercise Sheet 4: Sets

1. Consider the set $A \stackrel{\text{def}}{=} \{\{\},\{\{\}\},\{\{\{\}\}\}\}$. If $x \in A$, how many elements might x have?
2. Prove that if $A \subseteq B$ then $A \cup B = B$
3. Prove that if $A \subseteq A'$ and $B \subseteq B'$ then $A \times B \subseteq A' \times B'$
4. What can you say about sets A and B if you know that
   (a) $A \cup B = A$
   (b) $A \cap B = A$
   (c) $A - B = A$
   (d) $A \cap B = B \cap A$
   (e) $A - B = B - A$
5. Draw the Hasse diagram for the subset relation $\subseteq$ among the sets $A \stackrel{\text{def}}{=} \{2,4,6\}$, $B \stackrel{\text{def}}{=} \{2,6\}$, $C \stackrel{\text{def}}{=} \{4,6\}$, and $D \stackrel{\text{def}}{=} \{4,6,8\}$.
6. Is $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$ true for all sets A and B? Either prove it or give a counterexample.
7. Is $\mathcal{P}(A \cup B) = \mathcal{P}(A) \cup \mathcal{P}(B)$ true for all sets A and B? Either prove it or give a counterexample.
8. Draw pictures illustrating the following subsets of $\mathbb{R}^2$.
   (a) $\{(x,y) \mid y = x^2 - x - 2\}$
   (b) $\{(x,y) \mid y < x\}$
   (c) $\{(x,y) \mid (y > 0 \wedge y = x)\} \cup \{(2,y) \mid y > 1\} \cup \{(0,0)\}$

9. Let S be a set of students, R a set of college rooms, P a set of professors, and C a set of courses. Let $L \subseteq S \times R$ be the relation containing (s, r) if student s lives in room r. Let $E \subseteq S \times C$ be the relation containing (s, c) if student s is enrolled for course c. Let $T \subseteq C \times P$ be the relation containing (c, p) if course c is lectured by professor p. Describe the following relations.

   (a) E^{-1}

   (b) $L^{-1}; E$

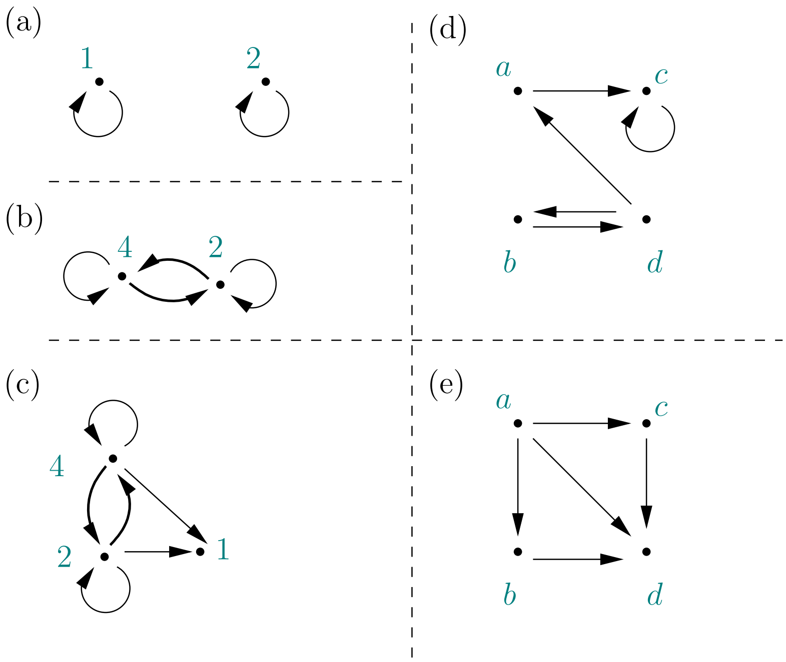
   (c) $E; E^{-1}$

   (d) $(L^{-1}; E); T$

   (e) $L^{-1}; (E; T)$

   (f) $(L^{-1}; L)^{+}$

10. For each of the following 5 relations, list its ordered pairs. Give a table showing for each whether it is reflexive, symmetric, transitive, acyclic, antisymmetric, and/or total.



11. Give a table showing, for each of the following relations over $\mathbb{N}$, whether it is reflexive, symmetric, transitive, or functional.

   (a) $n \ R \ m \stackrel{\text{def}}{=} n = 2m$

   (b) $n \ R \ m \stackrel{\text{def}}{=} 2n = m$

   (c) $n \ R \ m \stackrel{\text{def}}{=} \exists\, k.k \geq 2 \wedge k \text{ divides } n \wedge k \text{ divides } m$

12. (a) If R and S are directed acyclic graphs over a set A, is $R; S$? Either prove it or give a counterexample.

   (b) If R and S are directed acyclic graphs over a set A, is $R \cup S$? Either prove it or give a counterexample.

   (c) If R and S are directed acyclic graphs over a set A, is $R \cap S$? Either prove it or give a counterexample.

   (d) If R is a relation over a set A, can it be both symmetric and antisymmetric? Either give an example or prove it cannot.

Exercise Sheet 5: Inductive Proof

In all of the following, please state your induction hypothesis explicitly as a predicate.

1. Prove that, for all natural numbers n, $\sum_{i=1}^{n} i^2 = n(n+1)(2n+1)/6$.
2. Prove that, for all natural numbers x, m, and n, $x^{m+n} = x^m\, x^n$.
3. Prove that for all $n \geq 3$, if n distinct points on a circle are joined by consecutive order by straight lines, then the interior angles of the resulting polygon add up to $180(n-2)$ degrees.
4. Prove that, for any positive integer n, a $2^n \times 2^n$ square grid with any one square removed can be tiled with L-shaped pieces consisting of 3 squares.
5. Consider the following pair of ML function declarations:

```
fun takew p [] = []
  | takew p (x::xs) = if p x then x ::  takew p xs else []
fun dropw p [] = []
  | dropw p (x::xs) = if p x then dropw p xs else x::xs
```

   Prove `(takew p xs) @ (dropw p xs) = xs` using induction. (Assume that function `p` always terminates.) [Software Engineering II, 2001, p.2, q.9b]

6. Consider the following two ML functions:

```
 fun sumfiv [] = 0
   | sumfiv (x::xs) = 5*x + sumfiv xs
 fun summing z [] = z
   | summing z (x::xs) = summing (z + x) xs
```

   Prove that `sumfiv xs` is equal to `5 * summing 0 xs`. [Software Engineering II, 1999, p.2, q.9c]