Contextual Equivalence

$$M_1 \cong^{ctx} M_2 .$$

iff

$\forall$ programs $\mathcal{C}[-]$ with a hole $[-]$ where it makes sense to plug both M_1 and M_2 (*):

if $\mathcal{C}[M_1] \Downarrow v$ then $\mathcal{C}[M_2] \Downarrow v$

and vice versa.

(*) producing an observable output

Denotational Semantics

- $[\![\tau]\!]$ are domains.
- $M \in PCF_\tau$ (i.e. $\vdash M : \tau$)
  $[\![M]\!] \in [\![\tau]\!]$

for closed terms or programs

$$\frac{\Gamma[x \mapsto \tau] \vdash M : \tau'}{\Gamma \vdash \underline{fn}\, x:\tau.\, M : \tau \to \tau'}$$

$$[\![\underline{fn}\, x:\tau.\, M]\!] = \cdots \lambda \cdots [\![\Gamma[x \mapsto \tau] \vdash M]\!]$$

for open terms:

$$[\![\Gamma \vdash M : \tau]\!] : [\![\Gamma]\!] \to [\![\tau]\!]$$

where $[\![\Gamma]\!]$ is a domain and the function is continuous.

Compositionality

$$[\![M]\!] = [\![M']\!]$$

$$\Rightarrow [\![C[M]]\!] = f_C [\![M]\!]$$ by compositionality

$$= f_C [\![M']\!]$$

$$= [\![C[M']]\!]$$

$\mathcal{C}[M_1] \Downarrow_\gamma V$

$\Rightarrow [\![\mathcal{C}[M_1]]\!] = [\![V]\!]$

$\Rightarrow [\![\mathcal{C}[M_2]]\!] = [\![V]\!]$

$\Rightarrow \mathcal{C}[M_2] \Downarrow_\gamma V$

Contexts

$$[\![x_1 \mapsto \tau_1, \ldots, x_n \mapsto \tau_n]\!]$$
$$= [\![\tau_1]\!] \times [\![\tau_2]\!] \times \cdots \times [\![\tau_n]\!]$$

$$[\![\emptyset]\!] = \{\perp\} = \emptyset_\perp$$

Terms

$$\Gamma \vdash 0 : \underline{nat}$$

$$[\![\Gamma \vdash 0]\!] : [\![\Gamma]\!] \longrightarrow [\![nat]\!] = \mathbb{N}_\perp$$

$$[\![x_1 \mapsto \tau_1, \ldots, x_n \mapsto \tau_n \vdash x_i : \tau_i]\!]$$
$$: [\![\tau_1]\!] \times \cdots \times [\![\tau_n]\!] \longrightarrow [\![\tau_i]\!]$$

$$[\![x_1 \mapsto \tau_1, \ldots, x_n \mapsto \tau_n \vdash x_i : \tau_i]\!]$$
$$(f_1, \ldots, f_n) = f_i$$

$$\frac{\Gamma \vdash M_1 : \tau' \to \tau \qquad \Gamma \vdash M_2 : \tau'}{\Gamma \vdash M_1\, M_2 : \tau}$$

$[\![\Gamma \vdash M_1\, M_2]\!] : [\![\Gamma]\!] \longrightarrow [\![\tau]\!]$

define it in terms of

$[\![\Gamma \vdash M_1]\!] : [\![\Gamma]\!] \longrightarrow ([\![\tau']\!] \to [\![\tau]\!])$

$[\![\Gamma \vdash M_2]\!] : [\![\Gamma]\!] \longrightarrow [\![\tau']\!]$

$[\![\Gamma \vdash M_1\, M_2]\!](\rho)$

$= \left([\![\Gamma \vdash M_1]\!]\,\rho\right)\left([\![\Gamma \vdash M_2]\!]\,\rho\right)$

$[\![\Gamma \vdash M_1]\!](\rho) \in ([\![\tau']\!] \to [\![\tau]\!])$

$[\![\Gamma \vdash M_2]\!](\rho) \in [\![\tau']\!]$

$$\frac{\Gamma[x \mapsto \tau] \vdash M : \tau'}{\Gamma \vdash \underline{fn}\ x : \tau .\ M : \tau \to \tau'}$$

$[\![\Gamma \vdash \underline{fn}\ x : \tau . M]\!] : [\![\Gamma]\!] \to ([\![\tau]\!] \to [\![\tau']\!])$

So for $\rho \in [\![\Gamma]\!]$,

$[\![\Gamma \vdash \underline{fn}\ x : \tau . M]\!]\ \rho : [\![\tau]\!] \to [\![\tau']\!]$

We have:

$$[\![\Gamma[x \mapsto \tau] \vdash M : \tau']\!] : \underbrace{[\![\Gamma[x \mapsto \tau]]\!]}_{[\![\Gamma]\!] \times [\![\tau]\!]} \to [\![\tau']\!]$$

$= \lambda d \in [\![\tau]\!] .\ [\![\Gamma[x \mapsto \tau] \vdash M : \tau']\!] (\rho, d)$.