$(D \to E, \sqsubseteq)$

cont. functions.

$f \sqsubseteq g$ iff $\forall x.\ fx \sqsubseteq gx$.

$\perp_{D\to E} = \lambda x.\ \perp_E$

$f_0 \sqsubseteq f_1 \sqsubseteq \cdots \sqsubseteq f_n \sqsubseteq \ldots : D \to E$

$\underbrace{\bigsqcup_n f_n}_{\in\, D \to E} = \lambda d.\ \bigsqcup_n \underbrace{f_n(d)}_{\text{in a chain}}$

- monotone ✓
- preserves lubs of chains

$$\left(\bigsqcup_n f_n\right)\left(\bigsqcup_k x_k\right) \stackrel{?}{=} \bigsqcup_k \left(\bigsqcup_n f_n\right)(x_k)$$

$$= \bigsqcup_n f_n\left(\bigsqcup_k x_k\right) \qquad\qquad = \bigsqcup_k \bigsqcup_n f_n(x_k)$$

$$= \bigsqcup_n \bigsqcup_k f_n(x_k)$$

$=$ Diagonal argument.

- Lub property

$$f_k \stackrel{?}{\sqsubseteq} \bigsqcup_n f_n \quad \forall k$$

$$\Downarrow \quad \forall x \;\; f_k(x) \sqsubseteq \left(\bigsqcup_n f_n\right) x$$

$$= \bigsqcup_n f_n(x) \quad ✓$$

Let $g : D \to E$

$$f_k \sqsubseteq g \quad \forall k \qquad (*)$$

Need to show that

$$\bigsqcup_k f_k \overset{?}{\sqsubseteq} g$$

iff $\forall x \quad \underbrace{(\bigsqcup_k f_k)(x)}_{= \bigsqcup_k f_k(x)} \sqsubseteq g(x)$

To show $\bigsqcup_k f_k(x) \sqsubseteq g(x)$

it suffices to show

$$\forall k \quad f_k(x) \sqsubseteq g(x)$$

and this is the case by $(*)$

Continuity of $\underline{\text{fix}}$:

- $\underline{\text{monotonicity}}$:

$$f \sqsubseteq g \Rightarrow \underline{\text{fix}}(f) \sqsubseteq \underline{\text{fix}}(g)$$

$\boxed{\text{Assume } f \sqsubseteq g}$

$$\frac{}{f(\underline{\text{fix}}\, f) \sqsubseteq \underline{\text{fix}}\, f} \qquad \frac{f(x) \sqsubseteq x}{\underline{\text{fix}}(f) \sqsubseteq x}$$

$$\frac{\dfrac{\overline{f \sqsubseteq g}}{f(\underline{\text{fix}}\, g) \sqsubseteq g(\underline{\text{fix}}\, g)} \qquad \overline{g(\underline{\text{fix}}\, g) \sqsubseteq \underline{\text{fix}}\, g}}{\dfrac{f(\underline{\text{fix}}\, g) \sqsubseteq \underline{\text{fix}}\, g}{\underline{\text{fix}}(f) \sqsubseteq \underline{\text{fix}}(g)}}$$

- $$\underline{\mathit{fix}}\left(\bigsqcup_n f_n\right) \stackrel{?}{\sqsubseteq} \bigsqcup_n \underline{\mathit{fix}}(f_n)$$

For monotone f it is always the case that $\bigsqcup_n f(x_n) \sqsubseteq f(\bigsqcup_n x_n)$

$$\bigsqcup_\ell f_\ell(\underline{\mathit{fix}}\, f_\ell)$$
$$\shortparallel$$
$$\bigsqcup_k \bigsqcup_n f_k(\underline{\mathit{fix}}\, f_n)$$
$$\shortparallel$$
$$\bigsqcup_k f_k\left(\bigsqcup_n \underline{\mathit{fix}}(f_n)\right)$$
$$\shortparallel$$
$$\frac{\left(\bigsqcup_k f_k\right)\left(\bigsqcup_n \underline{\mathit{fix}}(f_n)\right) \sqsubseteq \bigsqcup_n \underline{\mathit{fix}}(f_n)}{\underline{\mathit{fix}}\left(\bigsqcup_n f_n\right) \sqsubseteq \bigsqcup_n \underline{\mathit{fix}}(f_n)}$$

$$\frac{f_k(\underline{fix}\ f_k) \sqsubseteq \underline{fix}\ f_k}{\bigsqcup_k f_k(\underline{fix}\ f_k) \sqsubseteq \bigsqcup_k \underline{fix}(f_k)}$$

Exercise:

$$\underline{fix}(f) = \bigsqcup_n f^n(\bot) \quad (*)$$

PROVE $\underline{fix}$ continuous using $(*)$.

We have

$$\underline{fix}(f) = \bigsqcup_n f^n(\bot)$$

and sometimes we do proofs concerning $f^n(\bot)$ by induction on n.

$f : D \to D$

$S \subseteq D \rightsquigarrow$ property we are interested in

SCOTT INDUCTION

$$\frac{\forall d \in D.\ d \in S \Rightarrow f(d) \in S}{\underline{fix}(f) \in S} \qquad \begin{array}{l} f \text{ cont.} \\ S \text{ admissible} \end{array}$$

$$\underline{fix}\, f = \bigsqcup_n f^n(\bot) \in S$$

1 $\boxed{\bot \in S} \Rightarrow f\bot \in S \Rightarrow f^2\bot \in S$

by ind. on $n \Rightarrow f^n(\bot) \in S$

2 $$\boxed{\begin{array}{c} \bot \sqsubseteq f\bot \sqsubseteq f^2(\bot) \sqsubseteq \dots \sqsubseteq f^n\bot \sqsubseteq \dots \in S \\ \Rightarrow \bigsqcup_n f^n\bot \in S \end{array}}$$

ADMISSIBILITY

$S \subseteq D$ is admissible

iff def

(1) $\bot \in S$

(2) $\forall\, x_0 \sqsubseteq x_1 \sqsubseteq \cdots \sqsubseteq x_n \sqsubseteq \cdots$ in S,

$\bigsqcup_n x_n \in S$

chain closure

$\forall f : D \to D \,.\, \forall S \subseteq D$ admissible

$(\forall d \in D.\ d \in S \Rightarrow f\,d \in S)$

$\Rightarrow \mathrm{fix}(f) \in S$

- $\sqsubseteq$ is admissible

* $(\bot,\bot) \in \sqsubseteq$ ✓

* $(x_0,y_0) \sqsubseteq (x_1,y_1) \sqsubseteq \cdots \sqsubseteq (x_n,y_n) \sqsubseteq \cdots$

  If $x_0 \sqsubseteq y_0,\ x_1 \sqsubseteq y_1, \ldots, x_n \sqsubseteq y_n, \ldots$

  Then $\bigsqcup_n x_n \sqsubseteq \bigsqcup_n y_n$ ✓

Let x be arbitrary such that

$x \sqsubseteq d \Rightarrow f(x) \sqsubseteq f(d) \sqsubseteq d$

← big assumption.

$$\frac{\equiv \forall x.\ x \sqsubseteq d \Rightarrow f(x) \sqsubseteq d \qquad \forall x \in {\downarrow}(d).\ f(x) \in {\downarrow}(d)}{\mathit{fix}(f) \sqsubseteq d \ \left(\equiv \mathit{fix}(f) \in {\downarrow}(d)\right)}$$