$[\![\underline{\text{while}}\ B\ \underline{\text{do}}\ C]\!]$

$$= \underline{\text{fix}}\Big(\overbrace{\lambda w : (\text{State} \rightharpoonup \text{State}) \to (\text{State} \rightharpoonup \text{State}).}^{f_{[\![B]\!],[\![C]\!]}}$$
$$\lambda s : \text{State}.$$
$$\underline{\text{if}}\big([\![B]\!](s),\ w([\![C]\!]s),\ s\big)\Big)$$

We want to $\underline{\text{approximate}}$ $[\![\underline{\text{while}}\ B\ \underline{\text{do}}\ C]\!]$

$\text{State} \rightharpoonup \text{States}$

(1) $\bot$ ~ the partial function with empty graph

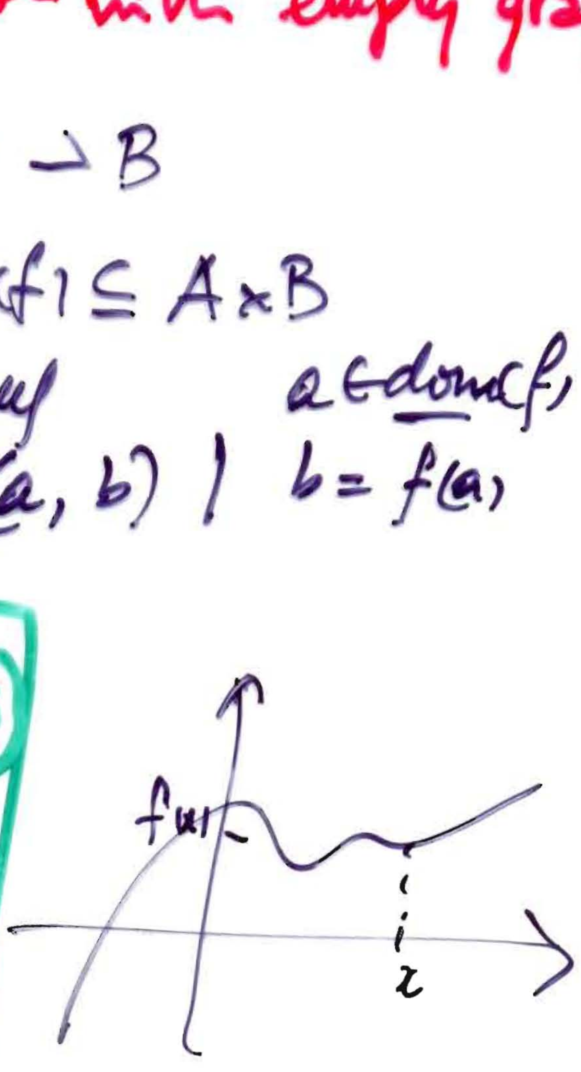
(2)

$$f_{[\![B]\!],[\![C]\!]}(\bot)$$
$$\|$$
$$\lambda s.\ \underline{\text{if}}\big([\![B]\!]s,\ \bot([\![C]\!]s),\ s\big)$$
$$\|$$
$$\lambda s.\ \underline{\text{if}}\big([\![B]\!]s,\ \uparrow,\ s\big)$$

$f : A \rightharpoonup B$

$\underline{\text{graph}}(f) \subseteq A \times B$

$$\overset{\text{def}}{=} \{(a, b) \mid a \in \underline{\text{dom}}(f),\ b = f(a)\}$$



(3) $f_{[\![B]\!],[\![C]\!]}(f_{[\![B]\!],[\![C]\!]}(\bot))$

$$= f_{[\![B]\!],[\![C]\!]}(\lambda s.\, \mathrm{if}([\![B]\!]s, \uparrow, s))$$

$$\begin{aligned} = \lambda s.\, \mathrm{if}(&[\![B]\!]s, \\ &(\lambda s.\, \mathrm{if}([\![B]\!]s, \uparrow, s))([\![C]\!]s), \\ &s) \end{aligned}$$

$$\begin{aligned} = \lambda s.\, \mathrm{if}(&[\![B]\!]s, \\ &\mathrm{if}([\![B]\!]([\![C]\!]s), \\ &\quad \uparrow, \\ &\quad [\![C]\!](s)), \\ &s) \end{aligned}$$

$$(n+1) \quad f_{[\![B]\!],[\![C]\!]}{}^{n}(\bot)$$

$$\underline{fix}(f_{[\![B]\!],[\![C]\!]}) \quad \text{is a } \underline{\text{least}} \text{ fixed point.}$$

$$=^{df} \bigcup_{n} f_{[\![B]\!],[\![C]\!]}{}^{n}(\bot).$$

D has a partial order structure

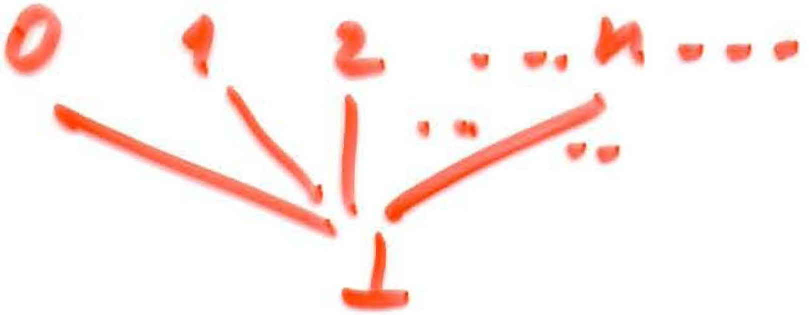
$$\sqsubseteq \; \subseteq D \times D$$

(R) $x \sqsubseteq x$

(T) $x \sqsubseteq y \wedge y \sqsubseteq z \Rightarrow x \sqsubseteq z$

(A) $x \sqsubseteq y \wedge y \sqsubseteq x \Rightarrow x = y$

$$C, s \Downarrow s' \quad \text{iff} \quad [\![C]\!](s) = s'$$

$\mathbb{N}_\perp =$ 

$$\mathbb{N}_\perp \xrightarrow{f} \mathbb{N}_\perp$$

$$\perp \longmapsto k \in \mathbb{N}.$$

$$n \longmapsto k+1 \in \mathbb{N}$$

not computable

Monotone

$$x \sqsubseteq y \Rightarrow f(x) \sqsubseteq f(y).$$

$\perp \sqsubseteq n$

but k is not $\sqsubseteq k+1$

$$\frac{}{x \le x}$$

$$\frac{x \le y \qquad y \le z}{x \le z}$$

$$\frac{x \le y \qquad y \le x}{x = y}$$

$$\frac{x \le y}{f(x) \le f(y)} \quad (f \text{ monotonic})$$

Least elements, if they exist, are unique.

$\mathbb{N} = 0 \ 1 \ 2 \ \ldots n \ldots$

Suppose d is least and also d' is least

$$\left.\begin{array}{l} d \sqsubseteq x \ \forall x \Rightarrow d \sqsubseteq d' \\ d' \sqsubseteq x \ \forall x \Rightarrow d' \sqsubseteq d \end{array}\right\} d = d'$$

$f : D \rightarrow D$

(1) PREFIXED POINT

$x \in D$ s.t. $f(x) \sqsubseteq x$

(2) $\underline{fix}(f)$ IF IT EXISTS is a prefixed pnt and is least amongst all other prefixed points

If d and d' are <u>least prefixed points</u> of f Then $d = d'$

(1) $\overset{(i)}{f(d) \sqsubseteq d}$ $\quad\quad$ $\overset{(ii)}{f(d') \sqsubseteq d'}$

(2) $\overset{(i)}{\forall x.}\ f(x) \sqsubseteq x \Rightarrow d \sqsubseteq x$

$\overset{(ii)}{\forall x.}\ f(x) \sqsubseteq x \Rightarrow d' \sqsubseteq x$

$$\left.\begin{array}{l} (1.i),(2\,ii) \Rightarrow d' \sqsubseteq d \\ (1\,ii),(2\,i) \Rightarrow d \sqsubseteq d' \end{array}\right\} \Longrightarrow d = d'$$

$$\frac{\quad}{f(fx(f))\ 5\ fx(f)}$$ (fo< f(x)(f), 4FP)

$$\frac{f(x)\ 5\ x}{fx(f)\ 5\ x}$$