

Adequacy

$$\llbracket M \rrbracket = \llbracket V \rrbracket \Rightarrow M \Downarrow V$$

Induction on M : ?

$$M \equiv M_1(M_2) : \tau$$

$$M_1 : \tau \rightarrow \tau$$

$$M_2 : \tau$$

$$\llbracket M_1(M_2) \rrbracket = \llbracket V \rrbracket \stackrel{?}{\Rightarrow} M_1(M_2) \Downarrow V$$

~~$\llbracket M_1 \rrbracket = \dots \Rightarrow M_1 \Downarrow \dots$~~

$$M = \text{fix}(M')$$

$$\begin{aligned} \text{fix}(M) \cap V &= \text{fix}(M) \cap V \stackrel{?}{\Rightarrow} \text{fix}(M) \cap V \\ \text{fix}(M) & \end{aligned}$$

$$\text{fix}(M) = \text{fix}(M) \rightarrow \text{fix}(M)$$

X

Logical relations

$$\begin{aligned} [M] \triangleleft M &\Rightarrow \text{adequacy.} \\ &\Downarrow \\ &\text{desiderata.} \end{aligned}$$

$$\Delta_{\text{not}} \subseteq N_{\perp} \times \underline{\text{PCF}}_{\text{not}}$$

$$\llbracket M_1(M_2) \rrbracket \triangleq_z M_1(M_2)$$

||

$$\llbracket M_1 \rrbracket (\llbracket M_2 \rrbracket)$$

by induction

$$\llbracket M_1 \rrbracket \triangleq_{z' \rightarrow z} M_1$$

$$\llbracket M_2 \rrbracket \triangleq_{z'} M_2$$

and would like to conclude

$$\llbracket M_1 \rrbracket (\llbracket M_2 \rrbracket) \triangleq_z M_1(M_2)$$

[?] How should we define

$$\Delta_{z' \rightarrow z} \subseteq (\Pi_{z'} \rightarrow \Pi_z) \times \underline{PCF}_{z' \rightarrow z}$$

?

$f \Delta_{z' \rightarrow z} M$

$\stackrel{\text{iff}}{\text{def}} \forall d \Delta_{z'} N$

$f(d) \Delta_z M(N)$

$$[\underline{\text{fix}}(M')] \stackrel{?}{\Delta} \underline{\text{fix}}(M')$$

$$\underline{\text{fix}}(\llbracket M' \rrbracket)$$

$$\llbracket M' \rrbracket : \llbracket z \rrbracket \rightarrow \llbracket z \rrbracket$$

$$\sqcup_n \llbracket M' \rrbracket^n (\perp)$$

$$(-) \Delta z N$$

are admissible.

Show

$$(1) \quad - \Delta_{\text{nat}} \underline{\text{succ}}^n(\underline{0}) \quad \forall n \in \mathbb{N}$$

$$- \Delta_{\text{bool}} \underline{\text{true}}$$

$$- \Delta_{\text{bool}} \underline{\text{false}}$$

admissible.

$$(2) \quad - \Delta_{z' \rightarrow z} N \text{ is admissible.}$$

$$\llbracket \underline{\text{fix}}(M') \rrbracket \triangleq \underline{\text{fix}}(M')$$

$$\llbracket M' \rrbracket \triangleq_{\text{c} \rightarrow \text{c}} M'$$

$$\frac{M'(\underline{\text{fix}} M') \Downarrow \checkmark}{\underline{\text{fix}}(M') \Downarrow \checkmark}$$

$$\begin{aligned} & \llbracket \underline{f_n} x.M \rrbracket \triangleq_{z' \rightarrow z} \underline{f_n} x.M \\ & \parallel \\ & \llbracket x \vdash M \rrbracket \end{aligned}$$

$$\begin{aligned} & \llbracket M \rrbracket \triangleq_M M \\ & M \in \underline{PCF}_z \end{aligned}$$

Generalise to a statement involving open terms.

$$\Gamma \equiv x_1 : \tau_1, \dots, x_n : \tau_n$$

$$\begin{aligned} & \llbracket \Gamma \vdash M \rrbracket \\ & : \llbracket \tau_1 \rrbracket \times \dots \times \llbracket \tau_n \rrbracket \rightarrow \llbracket \tau \rrbracket \end{aligned}$$

If $d_1 \triangleq_{z_1} M_1, \dots, d_n \triangleq_{z_n} M_n$

then

$$\begin{aligned} & \llbracket \Gamma \vdash M \rrbracket (d_1, \dots, d_n) \triangleq_z M \left[\overset{M_1}{x_1}, \dots, \overset{M_n}{x_n} \right] \\ & \quad \cap \quad \cap \\ & \llbracket \tau \rrbracket \quad \underline{PCF}_z \end{aligned}$$

Thm:

$$[M_1] \triangleleft M_2$$

$$\text{iff } M_1 \subseteq_{\text{Set}} M_2$$

$$M_1 \subseteq_{\text{Set}} M_2 : \tau_1 \rightarrow \tau_2 \rightarrow \dots \rightarrow \tau_n \rightarrow \delta^1$$

$$\text{iff } \forall n_1 \dots n_n$$

$$M_1, n_1, \dots, n_n \subseteq_{\text{Set}} M_2, n_1, \dots, n_n$$

$$\text{iff } \forall n_1 \dots n_n$$

$$M_1, n_1, \dots, n_n \Downarrow V \Rightarrow M_2, n_1, \dots, n_n \Downarrow V$$