Denotational Semantics

Syntactic categories ⟶ Abstract domains

$A \in \underline{Aexp}$

$\mathcal{A}[\![A]\!] : State \to \mathbb{Z}$

where $State = (\mathbb{L} \to \mathbb{Z})$.

$\mathcal{C}[\![C]\!]$ — is a state transformer

$: State \to State$.

$\mathcal{A}[\![L]\!] : State \to \mathbb{Z}$

$s \mapsto s(L)$ "look up"

(recall $State = \mathbb{L} \to \mathbb{Z}$)

$\mathcal{A}[\![A_1 + A_2]\!](s) = \mathcal{A}[\![A_1]\!](s) + \mathcal{A}[\![A_2]\!](s)$

$\mathcal{C}[\![C]\!] : State \to State$

$[\![L := A]\!] : State \to \underbrace{(\mathbb{L} \to \mathbb{Z})}_{= State.}$

$$[\![L := A]\!](s)(\ell) = \begin{cases} s(\ell) & \ell \neq L \\ [\![A]\!](s) & \ell = L \end{cases}$$

$$[\![c_1; c_2]\!](s) = [\![c_2]\!]([\![c_1]\!](s)).$$

Prop. $c, s \Downarrow s' \iff [\![c]\!](s) = s'$

$[\![\underline{\text{while}}\ B\ \underline{\text{do}}\ C]\!](s)$

$= ?$

$\underline{\text{while}}\ B\ \underline{\text{do}}\ C \equiv \underline{\text{if}}\ B\ \underline{\text{then}}\ C;\ \underline{\text{while}}\ B\ \underline{\text{do}}\ C\ \underline{\text{else}}\ \underline{\text{skip}}.$

$[\![\underline{\text{while}}\ B\ \underline{\text{do}}\ C]\!](s)$

$= [\![\underline{\text{if}}\ B\ \underline{\text{then}}\ C; \underline{\text{while}}\ B\ \underline{\text{do}}\ C\ \underline{\text{else}}\ \underline{\text{skip}}]\!](s)$

$$= \begin{cases} [\![\underline{\text{skip}}]\!](s) & , [\![B]\!](s) = \text{false} \\ [\![C; \underline{\text{while}}\ B\ \underline{\text{do}}\ C]\!](s) & , [\![B]\!](s) = \text{true} \end{cases}$$

$$= \begin{cases} s & , \text{if } [\![B]\!](s) = \text{false} \\ [\![\underline{\text{while}}\ B\ \underline{\text{do}}\ C]\!]([\![C]\!]s) & , \text{if } [\![B]\!](s) = \text{true} \end{cases}$$

cannot be made into a definition!

$[\![\underline{\text{while}}\ B\ \underline{\text{do}}\ C]\!]$ is a fixed point.

That is,

$$[\![\underline{\text{while}}\ B\ \underline{\text{do}}\ C]\!] = f_{\pi_B, \pi_C}([\![\underline{\text{while}}\ B\ \underline{\text{do}}\ C]\!])$$

$$f_{[\![B]\!],[\![C]\!]} = \lambda w.\lambda s.\begin{cases} s & \text{, if } [\![B]\!](s) = \text{false} \\ w([\![C]\!](s)) & \text{, if } [\![B]\!](s) = \text{true} \end{cases}$$

$$[\![\underline{\text{while}}\ B\ \underline{\text{do}}\ C]\!] = \underline{fix}(f_{[\![B]\!],[\![C]\!]})$$

↓

is computable

But what is $\underline{fix}$?