

Lecture 2

Least Fixed Points

Thesis

All domains of computation are partial orders with a least element.

All computable functions are monotonic.

Partially ordered sets

A binary relation \sqsubseteq on a set D is a **partial order** iff it is

reflexive: $\forall d \in D. d \sqsubseteq d$

transitive: $\forall d, d', d'' \in D. d \sqsubseteq d' \sqsubseteq d'' \Rightarrow d \sqsubseteq d''$

anti-symmetric: $\forall d, d' \in D. d \sqsubseteq d' \sqsubseteq d \Rightarrow d = d'$.

Such a pair (D, \sqsubseteq) is called a **partially ordered set**, or **poset**.

$$\overline{x \sqsubseteq x}$$

$$\frac{x \sqsubseteq y \quad y \sqsubseteq z}{x \sqsubseteq z}$$

$$\frac{x \sqsubseteq y \quad y \sqsubseteq x}{x = y}$$

Domain of partial functions, $X \rightarrow Y$

Underlying set: all partial functions, f , with domain of definition $\text{dom}(f) \subseteq X$ and taking values in Y .

Partial order:

$$\begin{aligned} f \sqsubseteq g & \text{ iff } \text{dom}(f) \subseteq \text{dom}(g) \text{ and} \\ & \forall x \in \text{dom}(f). f(x) = g(x) \\ & \text{ iff } \text{graph}(f) \subseteq \text{graph}(g) \end{aligned}$$

Monotonicity

- A function $f : D \rightarrow E$ between posets is **monotone** iff

$$\forall d, d' \in D. d \sqsubseteq d' \Rightarrow f(d) \sqsubseteq f(d').$$

$$\frac{x \sqsubseteq y}{f(x) \sqsubseteq f(y)} \quad (f \text{ monotone})$$

Least Elements

Suppose that D is a poset and that S is a subset of D .

An element $d \in S$ is the *least* element of S if it satisfies

$$\forall x \in S. d \sqsubseteq x .$$

- Note that because \sqsubseteq is anti-symmetric, S has at most one least element.
- Note also that a poset may not have least element.

Pre-fixed points

Let D be a poset and $f : D \rightarrow D$ be a function.

An element $d \in D$ is a **pre-fixed point of f** if it satisfies $f(d) \sqsubseteq d$.

The *least pre-fixed point* of f , if it exists, will be written

$$\boxed{fix(f)}$$

It is thus (uniquely) specified by the two properties:

$$f(fix(f)) \sqsubseteq fix(f) \quad (\text{lfp1})$$

$$\forall d \in D. f(d) \sqsubseteq d \Rightarrow fix(f) \sqsubseteq d. \quad (\text{lfp2})$$

Proof principle

1.

$$\frac{}{f(\text{fix}(f)) \sqsubseteq \text{fix}(f)}$$

2. Let D be a poset and let $f : D \rightarrow D$ be a function with a least pre-fixed point $\text{fix}(f) \in D$.

For all $x \in D$, to prove that $\text{fix}(f) \sqsubseteq x$ it is enough to establish that $f(x) \sqsubseteq x$.

$$\frac{f(x) \sqsubseteq x}{\text{fix}(f) \sqsubseteq x}$$

Least pre-fixed points are fixed points

If it exists, the least pre-fixed point of a mononote function on a partial order is necessarily a fixed point.

Thesis^{*}

All domains of computation are complete partial orders with a least element.

All computable functions are continuous.

Cpo's and domains

A **chain complete poset**, or **cpo** for short, is a poset (D, \sqsubseteq) in which all countable increasing chains $d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \dots$ have least upper bounds, $\bigsqcup_{n \geq 0} d_n$:

$$\forall m \geq 0 . d_m \sqsubseteq \bigsqcup_{n \geq 0} d_n \quad (\text{lub1})$$

$$\forall d \in D . (\forall m \geq 0 . d_m \sqsubseteq d) \Rightarrow \bigsqcup_{n \geq 0} d_n \sqsubseteq d. \quad (\text{lub2})$$

A **domain** is a cpo that possesses a least element, \perp :

$$\forall d \in D . \perp \sqsubseteq d.$$

$$\overline{\perp \sqsubseteq x}$$

$$\overline{x_i \sqsubseteq \bigsqcup_{n \geq 0} x_n} \quad (i \geq 0 \text{ and } \langle x_n \rangle \text{ a chain})$$

$$\frac{\forall n \geq 0 . x_n \sqsubseteq x}{\bigsqcup_{n \geq 0} x_n \sqsubseteq x} \quad (\langle x_i \rangle \text{ a chain})$$

Domain of partial functions, $X \rightarrow Y$

Underlying set: all partial functions, f , with domain of definition $dom(f) \subseteq X$ and taking values in Y .

Partial order:

$$\begin{aligned} f \sqsubseteq g \quad \text{iff} \quad & dom(f) \subseteq dom(g) \text{ and} \\ & \forall x \in dom(f). f(x) = g(x) \\ & \text{iff} \quad graph(f) \subseteq graph(g) \end{aligned}$$

Lub of chain $f_0 \sqsubseteq f_1 \sqsubseteq f_2 \sqsubseteq \dots$ is the partial function f with $dom(f) = \bigcup_{n \geq 0} dom(f_n)$ and

$$f(x) = \begin{cases} f_n(x) & \text{if } x \in dom(f_n), \text{ some } n \\ \text{undefined} & \text{otherwise} \end{cases}$$

Least element \perp is the totally undefined partial function.

Some properties of lubs of chains

Let D be a cpo.

1. For $d \in D$, $\bigsqcup_n d = d$.
2. For every chain $d_0 \sqsubseteq d_1 \sqsubseteq \dots \sqsubseteq d_n \sqsubseteq \dots$ in D ,

$$\bigsqcup_n d_n = \bigsqcup_n d_{N+n}$$

for all $N \in \mathbb{N}$.

3. For every pair of chains $d_0 \sqsubseteq d_1 \sqsubseteq \dots \sqsubseteq d_n \sqsubseteq \dots$ and $e_0 \sqsubseteq e_1 \sqsubseteq \dots \sqsubseteq e_n \sqsubseteq \dots$ in D ,
if $d_n \sqsubseteq e_n$ for all $n \in \mathbb{N}$ then $\bigsqcup_n d_n \sqsubseteq \bigsqcup_n e_n$.

$$\frac{\forall n \geq 0. x_n \sqsubseteq y_n}{\bigsqcup_n x_n \sqsubseteq \bigsqcup_n y_n} \quad (\langle x_n \rangle \text{ and } \langle y_n \rangle \text{ chains})$$

Diagonalising a double chain

Lemma. Let D be a cpo. Suppose that the doubly-indexed family of elements $d_{m,n} \in D$ ($m, n \geq 0$) satisfies

$$m \leq m' \ \& \ n \leq n' \Rightarrow d_{m,n} \sqsubseteq d_{m',n'}. \quad (\dagger)$$

Then

$$\bigsqcup_{n \geq 0} d_{0,n} \sqsubseteq \bigsqcup_{n \geq 0} d_{1,n} \sqsubseteq \bigsqcup_{n \geq 0} d_{2,n} \sqsubseteq \dots$$

and

$$\bigsqcup_{m \geq 0} d_{m,0} \sqsubseteq \bigsqcup_{m \geq 0} d_{m,1} \sqsubseteq \bigsqcup_{m \geq 0} d_{m,2} \sqsubseteq \dots$$

Moreover

$$\bigsqcup_{m \geq 0} \left(\bigsqcup_{n \geq 0} d_{m,n} \right) = \bigsqcup_{k \geq 0} d_{k,k} = \bigsqcup_{n \geq 0} \left(\bigsqcup_{m \geq 0} d_{m,n} \right).$$

Continuity and strictness

- If D and E are cpo's, the function f is **continuous** iff
 1. it is monotone, and
 2. it preserves lubs of chains, *i.e.* for all chains $d_0 \sqsubseteq d_1 \sqsubseteq \dots$ in D , it is the case that

$$f\left(\bigsqcup_{n \geq 0} d_n\right) = \bigsqcup_{n \geq 0} f(d_n) \quad \text{in } E.$$

- If D and E have least elements, then the function f is **strict** iff $f(\perp) = \perp$.

Tarski's Fixed Point Theorem

Let $f : D \rightarrow D$ be a continuous function on a domain D . Then

- f possesses a least pre-fixed point, given by

$$\text{fix}(f) = \bigsqcup_{n \geq 0} f^n(\perp).$$

- Moreover, $\text{fix}(f)$ is a fixed point of f , *i.e.* satisfies $f(\text{fix}(f)) = \text{fix}(f)$, and hence is the **least fixed point** of f .

[[while B do C]]

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$$= \text{fix}(f_{[[B]], [[C]])}$$

$$= \bigsqcup_{n \geq 0} f_{[[B]], [[C]]}^n(\perp)$$

$$= \lambda s \in \text{State}.$$

$$\left\{ \begin{array}{ll} [[C]]^k(s) & \text{if } k \geq 0 \text{ is such that } [[B]]([[C]]^k(s)) = \text{false} \\ & \text{and } [[B]]([[C]]^i(s)) = \text{true for all } 0 \leq i < k \\ \text{undefined} & \text{if } [[B]]([[C]]^i(s)) = \text{true for all } i \geq 0 \end{array} \right.$$