The Power of Random Bits

Randomized Algorithms: Applications & Principles

Part II: Random Routing and Load Balancing

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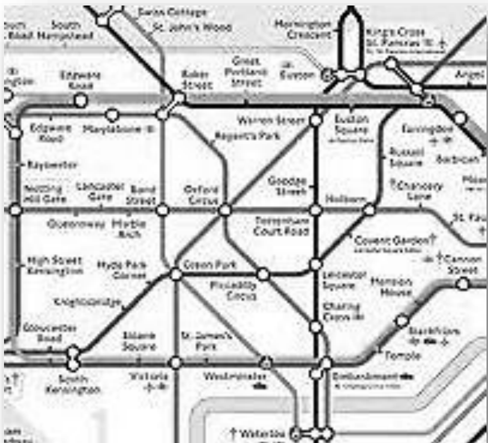
Problem: Traffic Routing

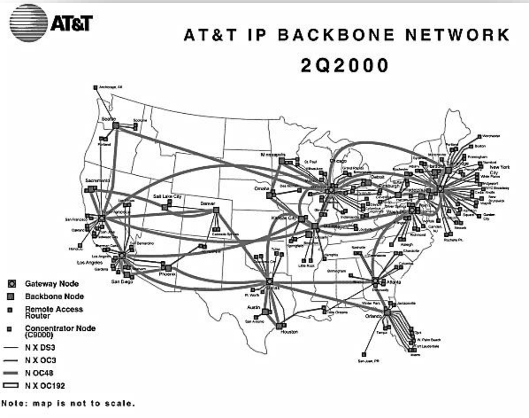


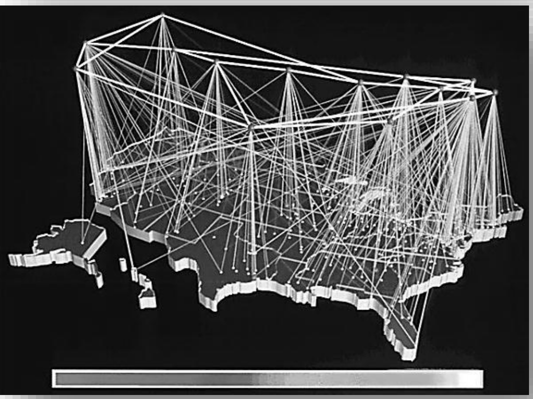
- Suppose you are in charge of transportation. What do you do to reduce congestion?
  - Congestion is caused by traffic demand exceeding the capacity of transport resource
  - To build more roads (to increase capacity)?
  - To raise toll (to reduce demand)?
  - Or to optimize the traffic routes and schedules (from algorithmic design)?
- Here is a radical idea – “random routing”:

1. A passenger wants to travel from a source to a destination
2. Take a passenger from the source to a “random” location
3. Then take the passenger from the “random” location to the destination

- Does this reduce congestion in transport networks?
- But this works in computer networks and telecommunication networks

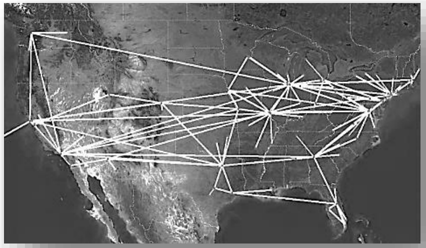
Random Routing in Tech Nets





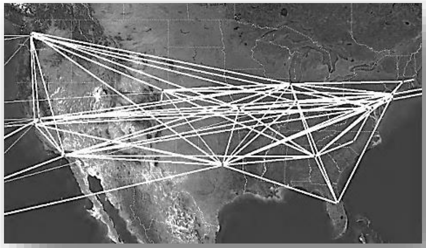
- Technological networks are interconnections of many nodes of systems and machines
  - High-performance supercomputers require intense communications among computing nodes (CPUs, GPUs, storage units)
  - Telecommunications need to forward numerous calls and data packets across places
- The connections are often sparse (as to reduce connection costs)
  - Require multihop relaying from nodes to nodes
- The nodes and links have limited I/O capacity
  - Unprocessed data are buffered in queues
- Congestion is caused by traffic demand exceeding network capacity at relays and links
- Random routing is implemented in these networks to reduce congestion and improve performance

Valiant Load Balancing



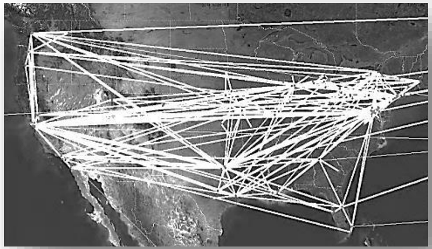
AT&T

80% utilization: 0.00008%
67% utilization: 0.09%



Sprint

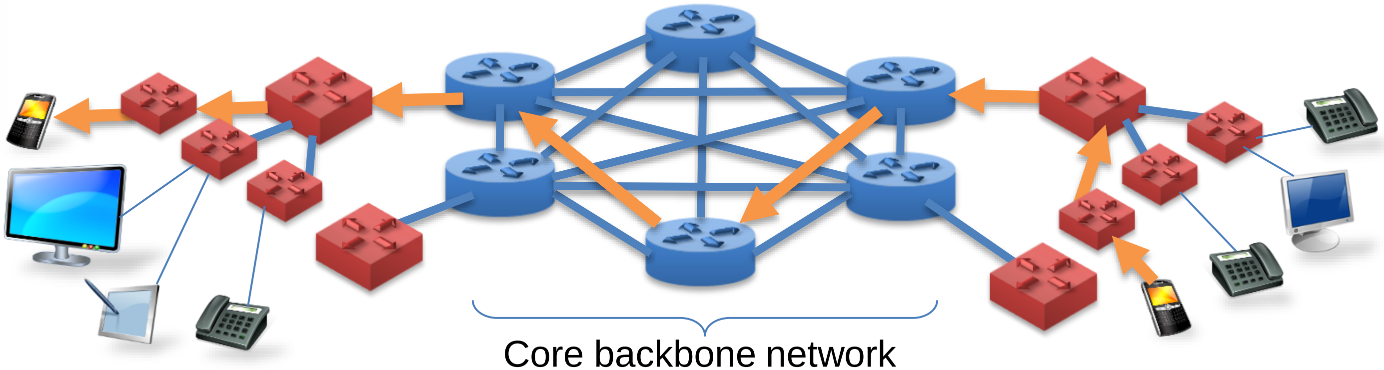
80% utilization: 0.0009%
67% utilization: 0.026%



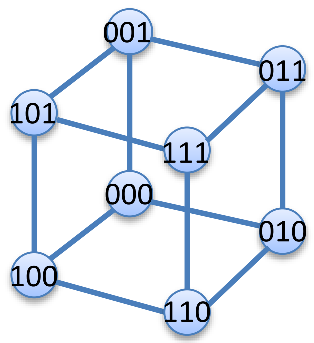
Verio

80% utilization: 0.0003%
67% utilization: 1.1%

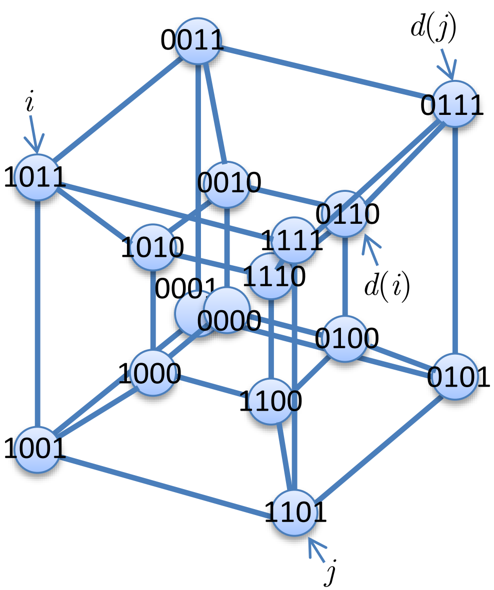
- Many Internet backbone networks are massively over-provisioned to provide reliable services
- Hence, the links are vastly underutilized
- How can we minimize the resource provision with satisfactory reliability?
- Valiant load balancing:
  - The core backbone network is a full-meshed network
  - Instead of the direct route between the source and destination, the route has to traverse a random intermediate router (i.e., random routing)
  - This balances the traffic among all routers in the core backbone network and averages out the utilization



Parallel Routing in Hypercube



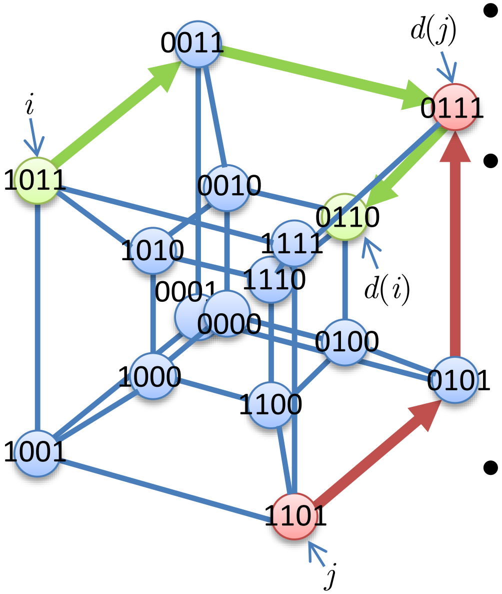
3-dimensional hypercube



4-dimensional hypercube

- Hypercube is an interconnection topology for supercomputers and peer-to-peer networks
- There are $N = 2^n$ nodes, each labelled by an n-bit coordinate
- There is a link between every pair of nodes with 1 bit difference in their coordinates
- Each link can transmit one packet at one time, and excessive packets will be buffered at nodes
- Assume that each node i has a destination $d(i)$, which may not necessarily be a neighbour (hence requiring multihop forwarding and buffering at relays)
- What is the minimum schedule of parallel routing (i.e., a sequence of sets of activated links) to forward the traffic from all the sources to destinations?
- Any simple algorithms? Computationally hard to find the minimum schedule by deterministic algorithms

Limit of Bit-Fixing Routing in Hypercube



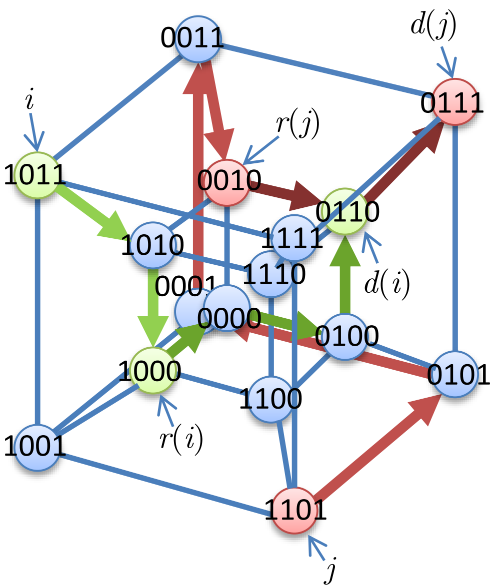
Bit-fixing routing

i	$\rightarrow$	$d(i)$
0000	$\rightarrow$	0000
0001	$\rightarrow$	0100
0010	$\rightarrow$	1000
0011	$\rightarrow$	1100
0100	$\rightarrow$	0001
0101	$\rightarrow$	0101
0111	$\rightarrow$	1101
1110	$\rightarrow$	1011
1111	$\rightarrow$	1111

A worst-case configuration

- A simple routing algorithm is oblivious to other flows -- find the shortest path between source and destination
- Bit-fixing routing is to find a path $(i_1, i_2, \ldots, d(i_1))$, where
  - (i_t, i_{t+1}) differ in only one bit for all t
  - if $(i_{t\text{-}1}, i_t)$ differ in the k-th leftmost bit and (i_t, i_{t+1}) differ in the l-th leftmost bit, then $k < l$
- There exists a configuration of sources and destinations that requires at least $2^{n/2}/2$ steps by bit-fixing routing
  - Consider n is even, for every source $i = (\mathtt{l}_i\, \mathtt{r}_i)$, we assign the destination to be $d(i) = (\mathtt{r}_i\, \mathtt{l}_i)$ (i.e., $d(i)$ is a transpose permutation of i)
  - Then for source $i = (?\ldots?1\ 0\ldots00)$ and its destination $d(i) = (0\ldots00\ ?\ldots?1)$ (i.e., $\mathtt{l}_i$ is odd and $\mathtt{r}_i$ is zero), it must traverse $(0\ldots01\ 0\ldots00)$ by bit-fixing routing
  - There are $2^{n/2}/2$ nodes with address $(?\ldots?1\ 0\ldots00)$
  - Only one source can traverse $(0\ldots01\ 0\ldots00)$ at one step
  - At least $2^{n/2}/2$ steps needed for relaying from these nodes

Random Routing in Hypercube



Random bit-fixing routing

i	$\rightarrow$	$r(i)$	$\rightarrow$	$d(i)$
0000	$\rightarrow$	*0000*	$\rightarrow$	0000
0001	$\rightarrow$	*0001*	$\rightarrow$	0100
0010	$\rightarrow$	*1000*	$\rightarrow$	1000
0011	$\rightarrow$	*0101*	$\rightarrow$	1100
0100	$\rightarrow$	*0001*	$\rightarrow$	0001
0101	$\rightarrow$	*1110*	$\rightarrow$	0101
0111	$\rightarrow$	*1101*	$\rightarrow$	1101
1110	$\rightarrow$	*0000*	$\rightarrow$	1011
1111	$\rightarrow$	*1110*	$\rightarrow$	1111

A two-stage configuration

- For deterministic bit-fix routing, the worst case requires at least $2^{n/2}/2$ steps (exponential in n)
- But for random bit-fix routing, it requires $\mathrm{O}(n)$ steps with high probability (i.e., using more than $\mathrm{O}(n)$ steps has a vanishing probability converging to 0, as $n \longrightarrow 0$)
- Random bit-fix routing has two stages:
  1. Pick a random node $r(i)$ in the hypercube independently, and use bit-fixing routing from i to $r(i)$
  2. Use bit-fixing routing from $r(i)$ to $d(i)$
- Obviously, longer paths are needed for random bit-fix routing. ***Then why is this better?***
- Intuition is that random routing can *average out* the worst case configuration from deterministic routing
- The probability that a randomly generated configuration is the worst case is very low, and is vanishing for large n
- This intuition is behind many randomized algorithms

Principle of Random Routing

- It suffices to show that it requires $\mathrm{O}(n)$ steps with high probability for the first stage of random bit-fixing routing
- For each source i, let P_i be the random path to a random node
- We observe a property of bit-fixing routing:
  - If P_i and P_j intersect, then there is only one subpath of intersection
  - P_i and P_j cannot intersect at multiple disjoint subpaths, as there is a unique path between any pair of nodes

- Let $\mathbf{1}(P_i, P_j)$ be the indicator function for testing if P_i and P_j intersect (once)
- The delay for source i is bounded by: $\text{delay}_i \le \sum_{j=1}^{2^n} \mathbf{1}(P_i, P_j)$
- Hence, the expected delay:

$$\mathbb{E}[\text{delay}_i] \le \mathbb{E}\left[\sum_{j=1:j\neq i}^{2^n} \mathbf{1}(P_i, P_j)\right] = \sum_{j=1:j\neq i}^{2^n} \mathbb{E}\left[\mathbf{1}(P_i, P_j)\right] \le \sum_{e\in P_i} \sum_{j=1:j\neq i}^{2^n} \mathbb{P}\{e \in P_j\}$$

where $e \in P_j$ denotes that e is a link in the path P_j

Continue in the next slide

Principle of Random Routing

Follow from the last slide

- Note that there are $n2^{n-1}$ links in a hypercube and 2^n paths by bit-fixing, where each path has at most n links
- Thus, the expected number of paths including a particular link e is 2: $\sum_{j=1}^{2^n} \mathbb{P}\{e \in P_j\} \leq 2$. Note that P_j contains at most n links
- Therefore, $\mathbb{E}[\text{delay}_i] \leq \sum_{j=1:j\neq i}^{2^n} \mathbb{E}\left[\mathbf{1}(P_i, P_j)\right] \leq 2n$
- Our aim is to show that $\mathbb{P}\left\{\sum_{j=1:j\neq i}^{2^n} \mathbf{1}(P_i, P_j) \geq cn\right\} \leq \frac{1}{2^n}$ for some c
- Hence, $\mathbb{P}\{\text{delay}_i \geq cn\} \leq \frac{1}{2^n}$ (i.e., it takes $\mathrm{O}(n)$ steps with high probability)
- We note that P_i and P_j are independent random variables (because $r(i)$ and $r(j)$ are picked independently)
    - So $\mathbf{1}(P_i, P_j)$ and $\mathbf{1}(P_i, P_k)$ are independent random variables for $i \neq j \neq k \neq i$
    - Let $X_j \triangleq \mathbf{1}(P_i, P_j)$ be a Bernoulli random variable: $\mathbb{P}\{X_j = 1\} = \mathbb{E}[X_j] \leq \frac{n}{n2^{n-1}}$
- Obtaining the distribution of sum of independent Bernoulli random variables, $\mathbb{P}\left\{\sum_{j=1}^{N} X_j \geq x\right\}$, requires *Chernoff Bound*

Summary

- Random routing takes a detour to a random intermediate node before reaching the destination
- Random routing can average out the worst case traffic patterns to deterministic routing algorithms
- Random routing has been implemented in telecommunication networks (Valiant load balancing) and in supercomputer architecture (parallel routing in hypercube)
- A key tool to prove the effectiveness of random routing is based on the Chernoff bound which estimates the exponential tail distribution of a sum of independent Bernoulli random variables
  - Hence, the probability that routing random deviates from the expected value is exponentially small in the size of network

References

- Main reference: Mitzenmacher and Upfal book, "*Probability and Computing: Randomized Algorithms and Probabilistic Analysis*"
  - Chapter 4.5: Packet routing in sparse networks
  - Chapter 4.2: Chernoff bound

- Additional reference
  - Rui Zhang-Shen and Nick McKeown, "*Designing a Predictable Internet Backbone with Valiant Load-Balancing*", Proceeding Workshop of QUALITY OF SERVICE (IWQoS) 2005