Linear Systems

Many of the systems that we wish to investigate are linear, time-invariant input-output systems.

That is, we put in a signal $f(t)$ and get out a signal $g(t)$

$$f(t) \longrightarrow \boxed{} \longrightarrow g(t)$$

"input-output" is obvious
"time-invarient" means the box doesn't change over time
so $f(t-T)$ in gives $g(t-T)$ out

Therefore, it is the "linear" bit that is important.

Linear systems are everywhere and are one of the most important and most exhaustively understood areas of continuous mathematics.

For the purposes of our discussion it is very important to know that "the eigenfunctions of linear systems are the complex exponentials". Put another way: "if you stick a sinusoid of frequency f into a linear system you got a sinusoid of frequency f out — the only thing it can change is the amplitude and phase offset of the sinusoid"

Now, any function $f(t)$ can be represented as the sum of weighted, offset sinusoids, so once you know how a linear system affects sinusoids, you know how it affects any function.

This is where Fourier analysis comes into play. It allows you to analyse and manipulate functions in terms of their frequencies rather than in terms of time or space.

Properties of linear systems

Proportionality: if the system's response to $f(t)$ is $g(t)$, then its response to $kf(t)$ will be $kg(t)$, k a constant.

Superposition: if the system's response to $f_1(t)$ is $g_1(t)$ and its response to $f_2(t)$ is $g_2(t)$ then its response to $f_1(t)+f_2(t)$ will be $g_1(t)+g_2(t)$

Linear systems can always be described by a linear operator, $h(t)$.

$$f(t) \longrightarrow \boxed{h(t)} \longrightarrow g(t)$$

$h(t)$ could be a convolution function, integration, differentiation, or some combination of the three.

$h(t)$ can be described by its Fourier transform, which will tell you what $h(t)$ will do to a sinusoid of any given frequency, and hence what it will do to <u>any</u> function, $f(t)$.

A closer look at the Fourier series

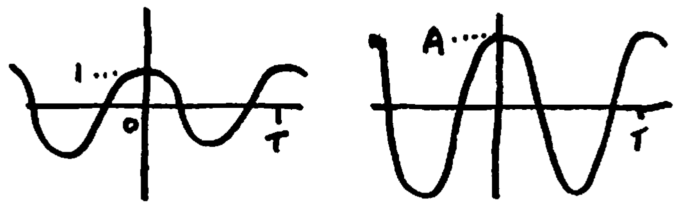
Any periodic function* can be represented as a Fourier series. There are three useful ways of looking at this series and, as you might have guessed, we will consider all three.

Take a function $f(x)$ with period T. That is:

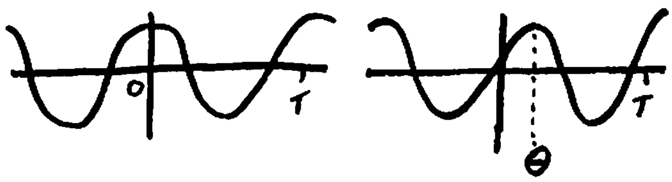
$$f(x+T) = f(x) \quad \forall x \qquad (F)$$

The Fourier series is a sum of weighted, offset sinusoids with frequencies that are multiples of $2\pi/T$. That is: sinusoids that have periods of $T, \frac{1}{2}T, \frac{1}{3}T, \ldots$ and which therefore also obey Equation (F) above.

Weighted means that we alter the amplitude



Offset means that we slide it along the x-axis



Neither affects the truth of Equation (F)

Thus, our first way of looking at the Fourier Series is:

$$f(x) = \sum_{k=0}^{\infty} A_k \cos\left(xk\frac{2\pi}{T} - \theta_k\right)$$

---

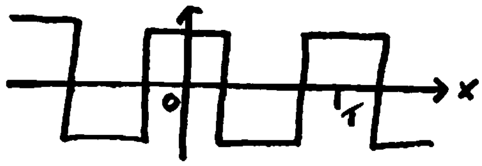
* not quite any: there are certain restrictions that you won't have to worry about for the purposes of this discussion

In this equation, A_k is the weight (scale factor), θ_k is the offset (phase) and the sinusoid has angular frequency $k\frac{2\pi}{T}$ which means that its wavelength is $\frac{T}{k}$.

Note that the series starts with a cosine of frequency zero. This is just a constant ($\cos 0 = 1$) and so θ_0 is irrelevant (why?)

An example

Consider the "square wave"



This can be represented as the Fourier series:

$$f(x) = \frac{4}{\pi}\cos\left(x\frac{2\pi}{T}\right) + \frac{4}{3\pi}\cos\left(x\cdot 3\frac{2\pi}{T} + \pi\right)$$
$$+ \frac{4}{5\pi}\cos\left(x\cdot 5\frac{2\pi}{T}\right) + \frac{4}{7\pi}\cos\left(x\cdot 7\frac{2\pi}{T} + \pi\right) + \ldots$$

That is:
$$A_k = \begin{cases} 0, & k \text{ even} \\ \frac{4}{k\pi}, & k \text{ odd} \end{cases}$$

$$\theta_k = \begin{cases} 0, & k \in \{1, 5, 9, 13, \ldots\} \\ \pi, & k \in \{3, 7, 11, 15, \ldots\} \\ \text{irrelevant}, & k \text{ even} \end{cases}$$

The supplementary sheets show this summation in operation.

[At this point you should be told that Fourier's colleagues thought that he was crazy to believe that you could construct a square wave out of a sum of sinusoids, but Fourier was right, so long as you sum all the way to the "infinityeth" term.]

Now, say you have

$$f(x) = \sum_{k=0}^{\infty} A_k \cos\left(x\, k \frac{2\pi}{T} - \theta_k\right)$$

What do you do if you want to scale the function?
That is, you want:

$$f'(x) = s.f(x)$$

This is mind-numingly simple:

$$A_k' = s.A_k$$

that is all there is to it!

Now, what if you want to shift the function along the x-axis by some distance d?
That is, you want:

$$f'(x) = f(x-d)$$

In this case you need to alter the phase terms:

$$\theta_k' = \theta_k + d k \frac{2\pi}{T}$$

Substitute this into the Fourier series equation to convince yourself that it has the desired effect.

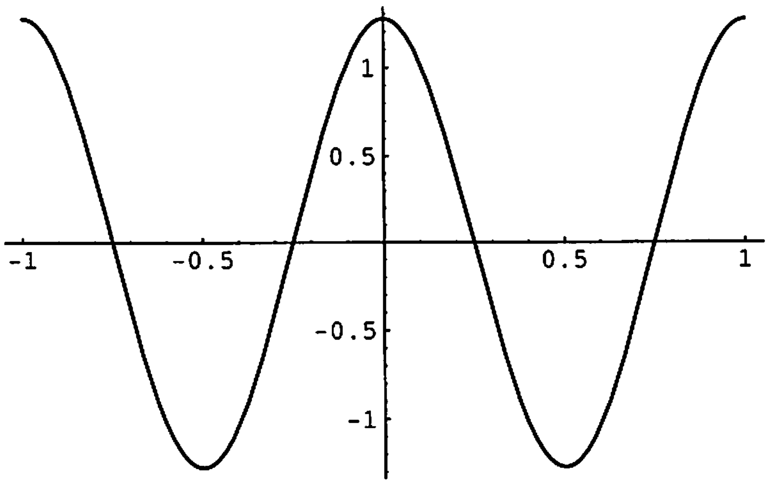
Finally, if you want to offset along the $f(x)$-axis, that is, you want $f'(x) = f(x) + D$, you need change only A_0:

$$A_0' = A_0 + D$$

(why is this so?)

(Standard Kernel) In[76]:=

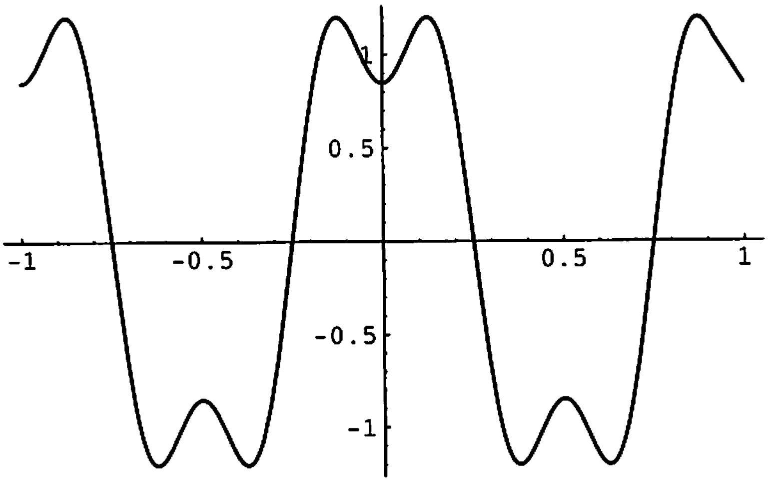
```
Plot[4/Pi * Cos[x*2Pi], {x,-1,1}]
```



The first term in the Fourier Series of a square wave.

(Standard Kernel) In[77]:=

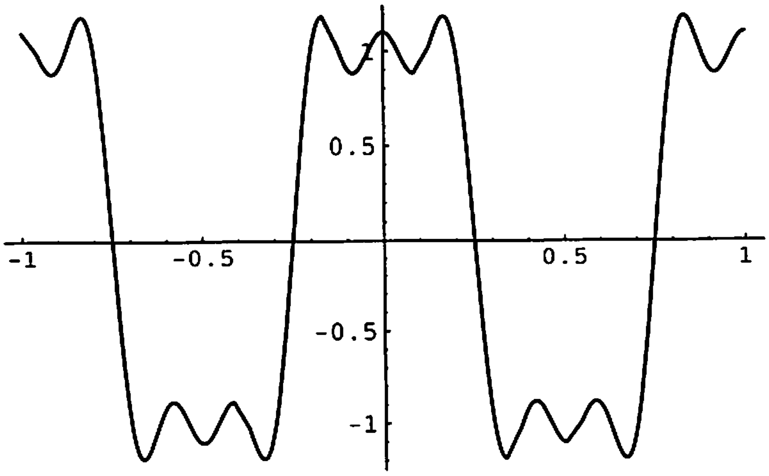
```
Plot[(4/Pi * Cos[x * 2Pi])
 + (4/3/Pi * Cos[x*3*2Pi + Pi]), {x,-1,1}]
```



The sum of the first two non-zero terms

(Standard Kernel) In[78]:=

```
Plot[(4/Pi * Cos[x * 2Pi])
 + (4/3/Pi * Cos[x*3*2Pi + Pi])
 + (4/5/Pi * Cos[x*5*2Pi]), {x,-1,1}]
```

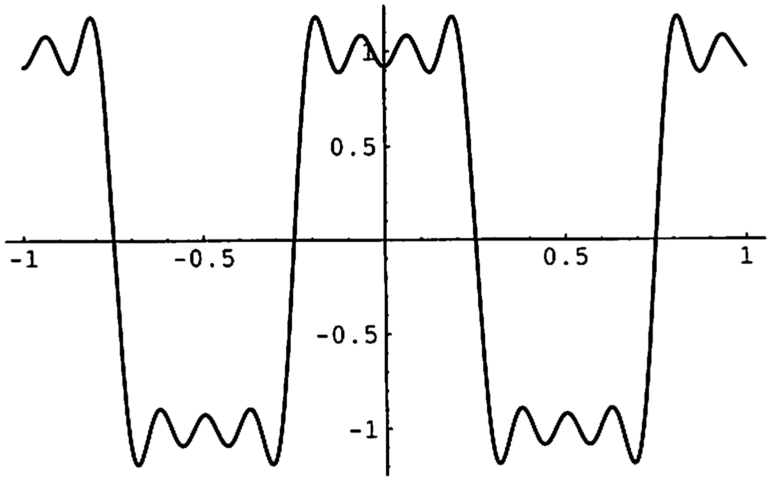


The sum of the first three non-zero terms

(Standard Kernel) In[79]:=

```
Plot[(4/Pi * Cos[x * 2Pi])
 + (4/3/Pi * Cos[x*3*2Pi + Pi])
 + (4/5/Pi * Cos[x*5*2Pi])
 + (4/7/Pi * Cos[x*7*2Pi + Pi]), {x,-1,1}]
```

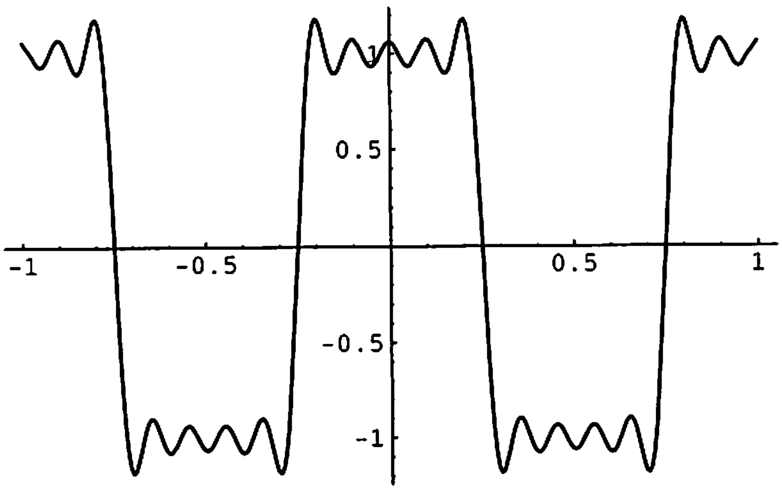
... the first four



(Standard Kernel) In[80]:=

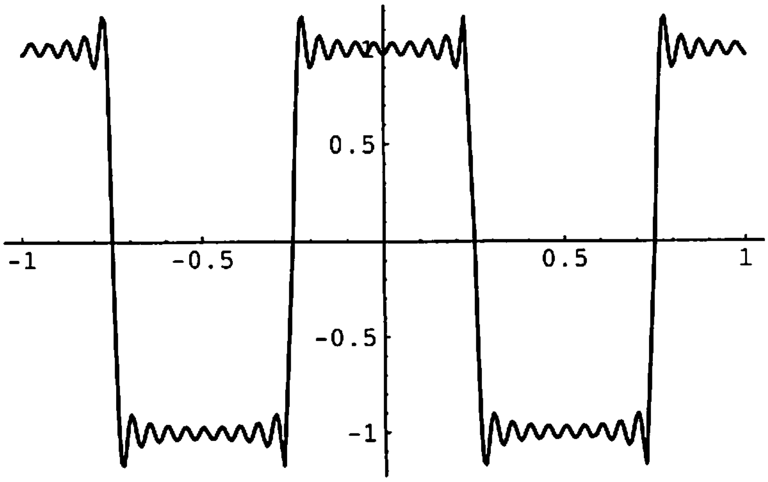
```
Plot[Sum[(4/k/Pi * Cos[x*k*2Pi + ((-1)^((k+1)/2)+1)/2*Pi]),
        {k,1,9,2}], {x,-1,1}, PlotPoints -> 10]
```

... the first five



(Standard Kernel) In[81]:=

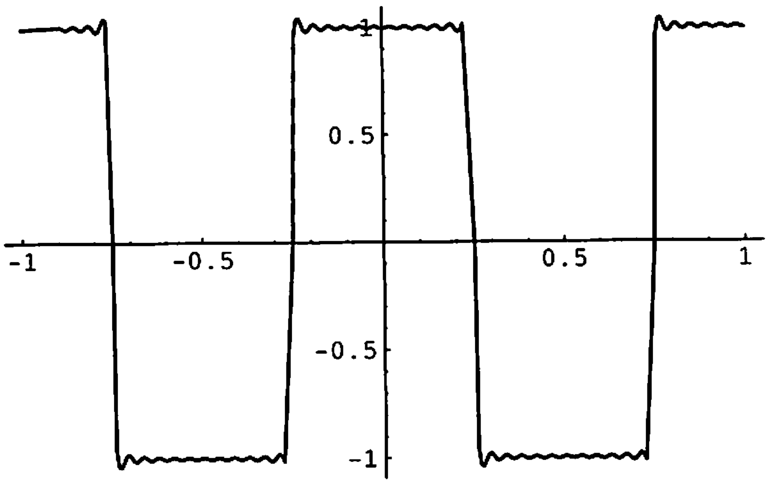
```
Plot[Sum[(4/k/Pi * Cos[x*k*2Pi + ((-1)^((k+1)/2)+1)/2*Pi]),
        {k,1,19,2}], {x,-1,1}, PlotPoints -> 10]
```



... the first ten

(Standard Kernel) In[82]:=

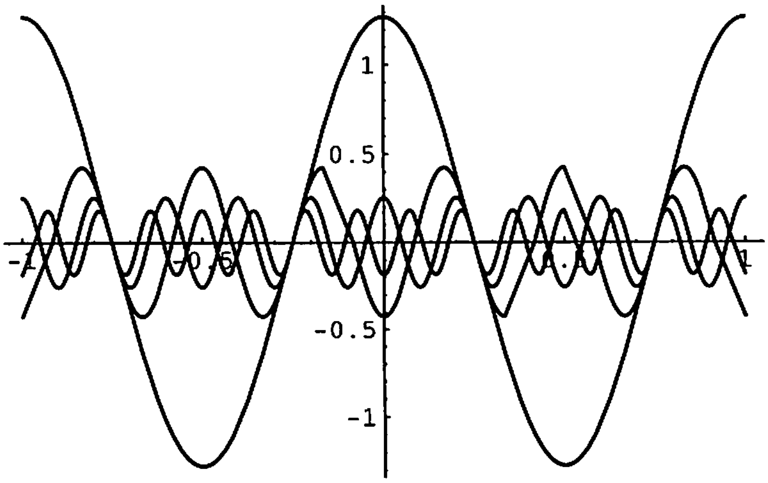
```
Plot[Sum[(4/k/Pi * Cos[x*k*2Pi + ((-1)^((k+1)/2)+1)/2*Pi]),
        {k,1,119,2}], {x,-1,1}, PlotPoints -> 10]
```



... the first sixty non-zero terms.

(Standard Kernel) In[83]:=

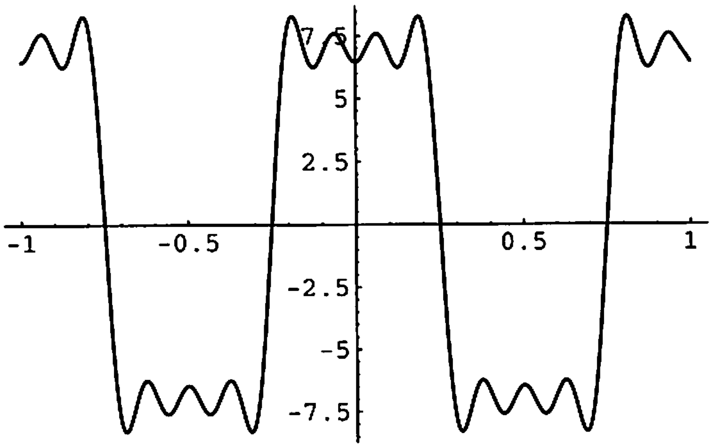
```
Plot[{(4/Pi * Cos[x * 2Pi]),
    (4/3/Pi * Cos[x*3*2Pi + Pi]),
    (4/5/Pi * Cos[x*5*2Pi]),
    (4/7/Pi * Cos[x*7*2Pi + Pi])}, {x,-1,1}]
```



The first four terms plotted individually so you can see the amplitudes and phases.

(Standard Kernel) In[84]:=

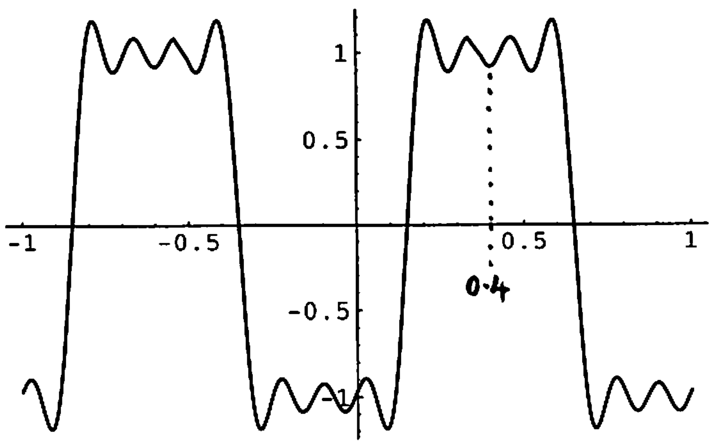
```
Plot[7.0 * Sum[(4/k/Pi * Cos[x*k*2Pi + ((-1)^((k+1)/2)+1)/2*Pi]),
        {k,1,7,2}], {x,-1,1}]
```



The first four non-zero terms summed

(Standard Kernel) In[85]:=

```
Plot[Sum[(4/k/Pi * Cos[x*k*2Pi + ((-1)^((k+1)/2)+1)/2*Pi - k*0.4*2Pi])),
        {k,1,7,2}], {x,-1,1}]
```

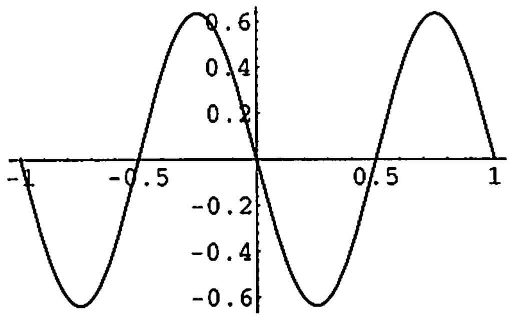


The same with a phase offset of:

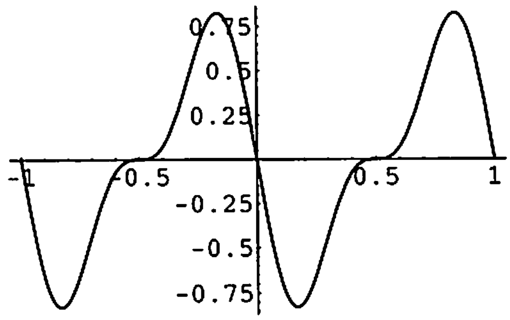
$+0.4 \times k \times 2\pi$

(Standard Kernel) In[69]:=

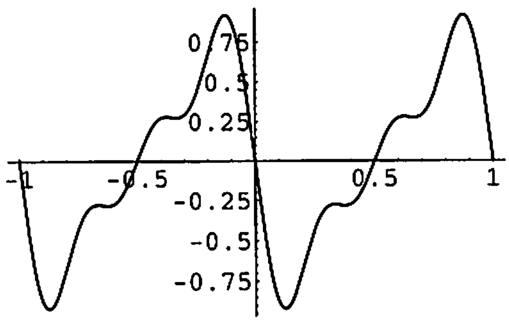
```
Plot[Sum[(2/k/Pi * Cos[x*k*2Pi + Pi/2]),
        {k,1,1,1}], {x,-1,1}]
Plot[Sum[(2/k/Pi * Cos[x*k*2Pi + Pi/2]),
        {k,1,2,1}], {x,-1,1}]
Plot[Sum[(2/k/Pi * Cos[x*k*2Pi + Pi/2]),
        {k,1,3,1}], {x,-1,1}]
Plot[Sum[(2/k/Pi * Cos[x*k*2Pi + Pi/2]),
        {k,1,4,1}], {x,-1,1}]
```



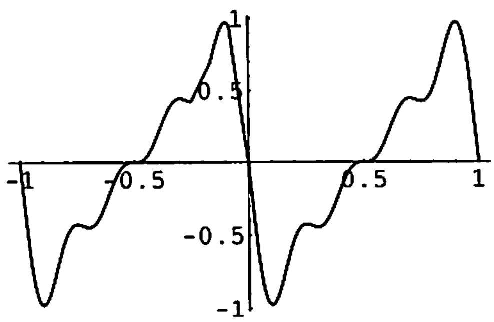
The first...



The first two...



The first three...

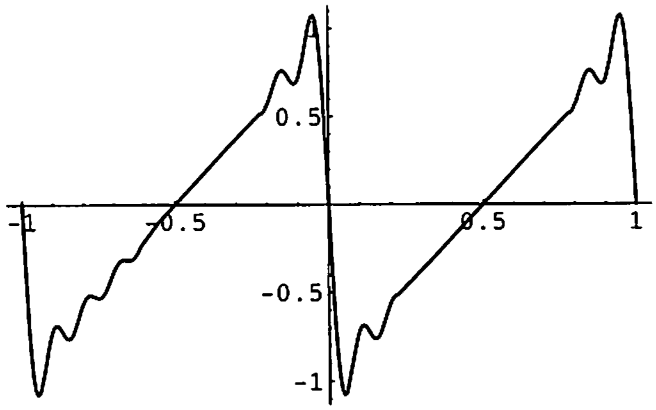


The first four...

non-zero terms in the Fourier series of a sawtooth

(Standard Kernel) In[73]:=

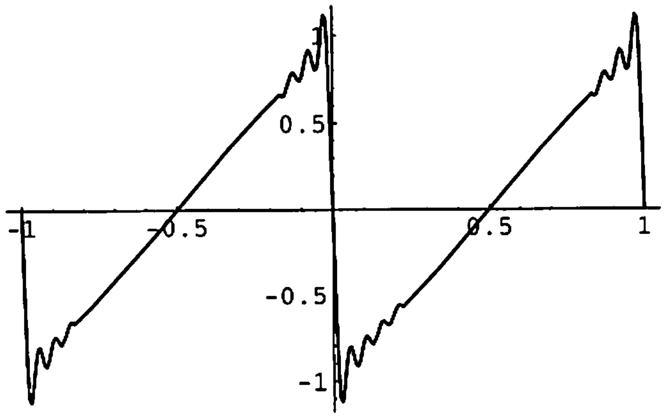
```
Plot[Sum[(2/k/Pi * Cos[x*k*2Pi + Pi/2]),
        {k,1,9,1}], {x,-1,1}, PlotPoints -> 10]
```



The first nine non-zero terms

(Standard Kernel) In[74]:=

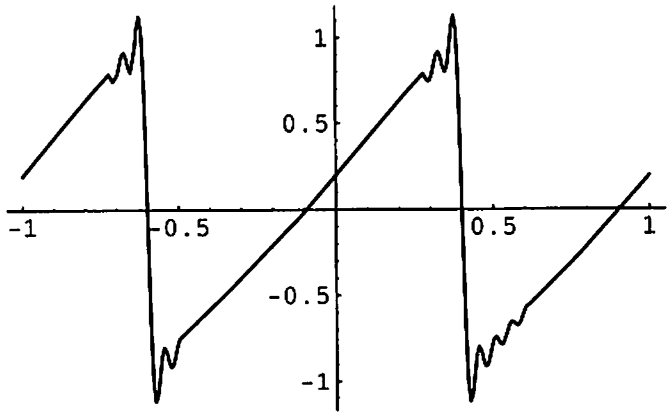
```
Plot[Sum[(2/k/Pi * Cos[x*k*2Pi + Pi/2]),
        {k,1,19,1}], {x,-1,1}, PlotPoints -> 10]
```



and the first nineteen

(Standard Kernel) In[75]:=

```
Plot[Sum[(2/k/Pi * Cos[x*k*2Pi + Pi/2 - k*0.4*2Pi]),
        {k,1,19,1}], {x,-1,1}, PlotPoints -> 10]
```



The same with a phase offset of 0·4 of a cycle (in this case $T=1$)

The supplementary sheets show these transformations being applied to our square wave. They also show some examples of the sawtooth wave, for which:

$$A_k = \begin{cases} 0, & k=0 \\ \frac{2}{k\pi}, & k>0 \end{cases}$$

$$\theta_k = -\frac{\pi}{2}, \quad \forall k$$

This is all very interesting but tells us nothing about how to compute the values of the A_k and θ_k.

It transpires that having θ_k inside the cosine argument is very nasty indeed when you try to perform serious mathematics on the Fourier series.

So we move to our second representation of the Fourier series. You will probably have met this before:

$$f(x) = a_0 + \sum_{k=1}^{\infty} a_k \cos\left(x \cdot k \frac{2\pi}{T}\right) + b_k \sin\left(x \cdot k \frac{2\pi}{T}\right)$$

The first thing to note is that we treat $k=0$ as a special case. This is because $\cos 0 = 1$ and $\sin 0 = 0$, therefore b_0 is irrelevant.

Furthermore, the formula to calculate a_0 is slightly different to the formulae for $k \geq 1$.

How does this second formulation of the Fourier series relate to the first?

For a given k we have the term:

first form: $A_k \cos\left(x \cdot k \frac{2\pi}{T} - \theta_k\right)$

second form: $a_k \cos\left(x \cdot k \frac{2\pi}{T}\right) + b_k \sin\left(x \cdot k \frac{2\pi}{T}\right)$

Now, recall that:

$$\cos(\mu - \phi) = \cos\mu \cos\phi + \sin\mu \sin\phi$$

so: $A_k \cos\left(x \cdot k \frac{2\pi}{T} - \theta_k\right)$

$$= A_k \cos\theta_k \cos\left(x \cdot k \frac{2\pi}{T}\right) + A_k \sin\theta_k \sin\left(x \cdot k \frac{2\pi}{T}\right)$$

and therefore:

$$a_k = A_k \cos\theta_k \qquad A_k = \sqrt{a_k^2 + b_k^2}$$

$$b_k = A_k \sin\theta_k \qquad \theta_k = \tan^{-1}\frac{b_k}{a_k}$$

Voila! We have got the nasty θ_k term out of the cosine's argument and things will run, mathematically, more smoothly.

Notes: (i) all we have done is encode the physical quantities, A_k and θ_k, as two more amenable values, a_k and b_k, we can always jump back to the first form when it is helpful to do so.

(ii) this looks awfully like complex numbers; we'll see the link when we get to the third form.

We are now in a position where we can calculate all those constants: a_k and b_k.

$$a_k = \frac{2}{T}\int_0^T f(x).\cos\left(x \cdot k\frac{2\pi}{T}\right)dx\,, \quad k \geq 1$$

$$b_k = \frac{2}{T}\int_0^T f(x).\sin\left(x \cdot k\frac{2\pi}{T}\right)dx\,, \quad k \geq 1$$

We've said that a_0 is a special case:

$$a_0 = \frac{1}{T}\int_0^T f(x).\cos\left(x \cdot 0\frac{2\pi}{T}\right)dx$$

$$= \frac{1}{T}\int_0^T f(x)\,dx$$

Note the extra factor of ½.

b_0, if we'd bothered to include it, would always be zero.

If you wish, you can now check whether or not I calculated the coefficients correctly for the square wave and sawtooth.

I believe that:

$$a_k = (-1)^{\frac{k-1}{2}} \cdot \frac{4}{k\pi}\,, \; k \text{ odd}$$

$$a_k = 0, \; k \text{ even}$$

$$b_k = 0, \; \forall k$$

SQUARE WAVE

$$a_k = 0, \quad \forall k$$

$$b_k = -\frac{2}{k\pi}\,, \quad \forall k \geq 1$$

SAWTOOTH

Now, the Fourier series is clever, but there is something clumsy about needing two coefficients and two different calculations for each frequency. It would be much better if we could package it up so that there were a consistent calculation for every frequency: no messing with both a cosine and a sine.

Initially, this may seem impossible, we need two coefficients at every frequency to encode magnitude (A_k) and phase (θ_k). No amount of fiddling can encode both in a single real number.

However, by using complex numbers and negative frequencies (!) we can produce the beautiful third form of the Fourier series:

$$f(x) = \sum_{k=-\infty}^{\infty} c_k \cdot e^{i x \cdot k \frac{2\pi}{T}}$$

Furthermore, we can calculate the complex coefficient, c_k, by the formula:

$$c_k = \frac{1}{T} \int_0^T f(x) . e^{-i . x . k \frac{2\pi}{T}} dx$$

How does this relate to the other forms of the Fourier series?

Let us consider the form of the basis functions, which are:

$$e^{i.x.k\frac{2\pi}{T}}$$

We know that $k \in \mathbb{Z}$ and that T is a constant, so

$$k\frac{2\pi}{T}$$

is, in some sense, the "frequency" of the function.*

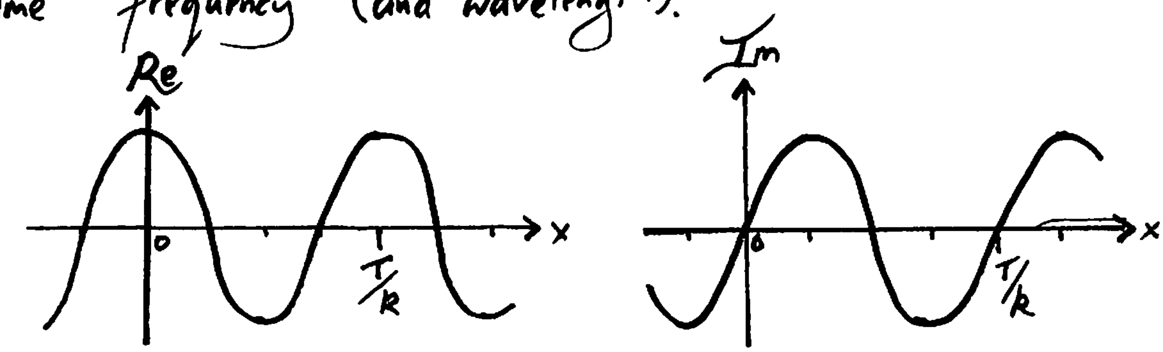
We also know that

$$e^{i\theta} = \cos\theta + i.\sin\theta$$

So:

$$e^{i.x.k\frac{2\pi}{T}} = \cos\left(k\frac{2\pi}{T}\cdot x\right) + i.\sin\left(k\frac{2\pi}{T}\cdot x\right)$$

The real part is a cosinusoid of frequency $k\frac{2\pi}{T}$ and wavelength T/k. The imaginery part is a sinusoid of the same frequency (and wavelength).



If we now plot this on the Argand diagram (three axes: x, Re, and Im) we get a curve which stays a constant distance from the x-axis and spirals around the axis like a corkscrew.

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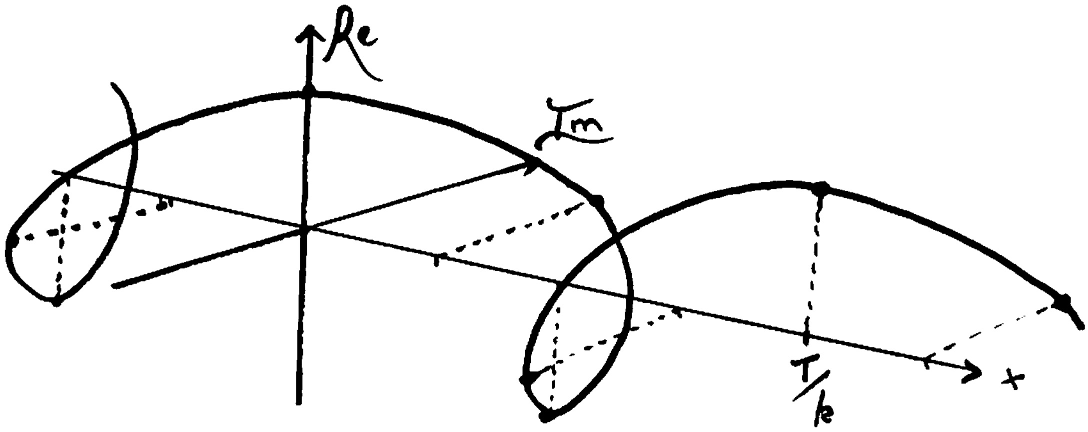
* it is the "angular frequency" expressed in radians



This may look odd but remember that, if you project it into the x-Re plane you get a cosine curve; if you project it into the x-Im plane you get a sine curve. So it is really pretty simple, after all.

Now, if $f(x)$ is a real-valued function then the c_k exhibit the property that the negative frequency co-efficients, c_{-k}, are the complex conjugates of the positive frequency co-efficients:

$$c_{-k} = c_k^*$$

Let $\quad c_k = v_k - i w_k \;, \quad k \geq 0$

then $\quad c_{-k} = v_k + i w_k \;, \quad k \geq 0$

To connect this back to our second form of the Fourier series we combine the negative and positive coefficients for each frequency, as shown on the following page.

You may wonder at the physical reality behind "negative" frequencies. It appears that, as with imaginery numbers, there is no coresponding physical reality — what they provide is a profoundly useful mathematical mechanism which allows us to easily analyse reality.

Take the positive and negative terms for a particular frequency:

$$c_k e^{i.x.k\frac{2\pi}{T}} + c_{-k} e^{-i.x.k\frac{2\pi}{T}} \quad , \; k>0$$

$$= c_k e^{i.x.k\frac{2\pi}{T}} + c_k^* e^{-i.x.k\frac{2\pi}{T}}$$

$$= (v_k - i w_k) e^{i.x.k\frac{2\pi}{T}} + (v_k + i w_k) e^{-i.x.k\frac{2\pi}{T}}$$

$$= (v_k - i w_k)\left(\cos\left[xk\tfrac{2\pi}{T}\right] + i.\sin\left[xk\tfrac{2\pi}{T}\right]\right)$$
$$+ (v_k + i w_k)\left(\cos\left[-xk\tfrac{2\pi}{T}\right] + i.\sin\left[-xk\tfrac{2\pi}{T}\right]\right)$$

$$= (v_k - i w_k)\left(\cos\left[xk\tfrac{2\pi}{T}\right] + i.\sin\left[xk\tfrac{2\pi}{T}\right]\right)$$
$$+ (v_k + i w_k)\left(\cos\left[xk\tfrac{2\pi}{T}\right] - i.\sin\left[xk\tfrac{2\pi}{T}\right]\right)$$

$$= (2v_k)\cos\left[xk\tfrac{2\pi}{T}\right] + (-2i^2 w_k)\sin\left[xk\tfrac{2\pi}{T}\right]$$

$$= 2v_k \cos\left[xk\tfrac{2\pi}{T}\right] + 2w_k \sin\left[xk\tfrac{2\pi}{T}\right].$$

Therefore: $a_k = 2v_k$ <u>and</u> $b_k = 2w_k$

So the amplitude and phase information are encoded by the two parts (v_i and w_i) of the complex coefficient c_k

$$A_k = 2\sqrt{v_k^2 + w_k^2} \qquad \theta_k = \tan^{-1}\frac{w_k}{v_k}$$

[Note that the complex numbers and negative frequencies all vanish when you do this manipulation → ☺]

Basis Functions, Orthogonality and Inner Products

We have, thus far, simply asserted that our Fourier series co-efficients can be calculated in a certain way. The reason that this is so lies in a more general piece of mathematical theory.

In general, almost any function, $f(x)$, can be represented as a linear combination of a family of <u>basis functions</u>:

$$f(x) = \sum_k a_k \Psi_k(x)$$

For the power series: $$\Psi_k(x) = x^k, \quad 0 \le k \le \infty$$

For the Fourier series: $$\Psi_k(x) = e^{ix.k\frac{2\pi}{T}}, \quad -\infty \le k \le \infty$$

Other, commonly used families of basis functions are the cosines (a variant of Fourier's complex exponentials), the "square wave" functions of Haar and Walsh-Hadamard, and various types of wavelet, including Gabor wavelets. You will meet the first two on the Computer Graphics & Image Processing course and the latter on the Information Theory & Coding and Computer Vision courses.

Whatever the family is, we need to find a way to calculate the a_k.

The concept of orthogonality of a family of functions proves useful here.

If the chosen family of basis functions satisfies the rule that the <u>inner product</u> of any two different members of the family equals zero

$$\langle \Psi_k(x), \Psi_j(x) \rangle = 0, \quad \forall_{j,k} : j \neq k$$

then the family of functions is called <u>orthogonal</u>.

The inner product is the integral of the conjugate product of the two functions:

$$\langle \Psi_k(x), \Psi_j(x) \rangle = \int_{-\infty}^{\infty} \Psi_k^*(x)\, \Psi_j(x)\, dx$$

If, in addition, the inner product of any member with itself is one:

$$\langle \Psi_k(x), \Psi_k(x) \rangle = 1 \quad \forall k$$

then the family is <u>orthonormal</u>.

If the family form a <u>complete orthonormal basis</u> then the a_k can be calculated as:

$$a_k = \langle \Psi_k(x), f(x) \rangle = \int_{-\infty}^{\infty} \Psi_k^*(x) f(x)\, dx$$

The Fourier series basis functions form a complete orthonormal basis so we can use the above formula.

The power series basis functions do <u>not</u> form a complete orthonormal basis, so their a_k must be calculated by some other method — which we derived earlier.

Complex Variations

If $f(x)$ is a complex valued function $f: \mathbb{R} \to \mathbb{C}$ then you cannot use the identity:

$$c_k = c_{-k}^*$$

because this is only true for real-valued functions.
However, you are unlikely to meet too many complex valued functions in your work so, unless explicitely stated otherwise, we will assume that we are working with real-valued functions, and that $c_k = c_{-k}^*$.

Non-periodic functions

So far, we have assumed that we have a continuous, periodic function, which can be represented by a Fourier series with discrete coefficients; each coefficient having its associated frequency.

In order to proceed any farther we need to consider the Fourier transform where the continuous function $f(x)$ is aperiodic and transforms to a continuous function of frequency $F(\omega)$.

This will then allow us to reach a deeper understanding of Fourier theory and, in particular, of how it relates to the finite-length sets of discrete samples that we deal with in digital computers.

Fourier Transforms

The Fourier transform of a function, $f(t)$, is:

$$F(\nu) = \int_{-\infty}^{\infty} f(t) e^{-i2\pi\nu t} dt$$

The inverse Fourier transform is defined as:

$$f(t) = \int_{-\infty}^{\infty} F(\nu) e^{i2\pi\nu t} dt$$

where ν is the frequency in Hertz (cycles per second).

You will meet people who prefer to use angular frequency, ω, measured in radians:

$$\omega = 2\pi\nu$$

this, unfortunately, leads to an extra constant factor of $1/2\pi$:

$$F(\omega) = a_1 \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

$$f(t) = a_2 \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega$$

$$a_1 a_2 = \frac{1}{2\pi}$$

You will see both: $a_1 = 1,\ a_2 = \frac{1}{2\pi}$ and $a_1 = a_2 = \frac{1}{\sqrt{2\pi}}$ in use.

We'll use the $F(\nu)$ form but be aware of the $F(\omega)$ form!

Example: the F.T. of a symmetrical rectangular pulse

$$f(t) = \begin{cases} A, & |t| < T_0 \\ A/2, & |t| = T_0 \\ 0, & |t| > T_0 \end{cases}$$

$$F(\nu) = \int_{-\infty}^{\infty} f(t)\, e^{-i2\pi\nu t}\, dt$$

$$= \int_{-T_0}^{T_0} A\, e^{-i2\pi\nu t}\, dt$$

$$= \frac{A}{-i2\pi\nu}\, e^{-i2\pi\nu t} \Big|_{-T_0}^{T_0}$$

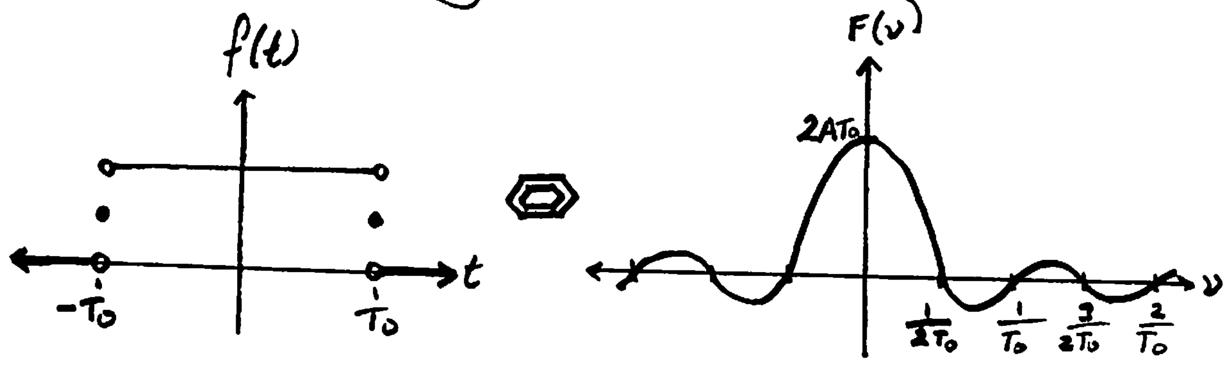
$$= \frac{A}{-i2\pi\nu}\left(e^{-i2\pi\nu T_0} - e^{i2\pi\nu T_0}\right)$$

$$= 2\frac{A}{2\pi\nu} \sin 2\pi\nu T_0$$

This can be rewritten:

$$F(\nu) = 2AT_0\, \frac{\sin(2\pi\nu T_0)}{2\pi\nu T_0}$$

The form $\frac{\sin ax}{ax}$ is called a <u>sinc</u> function and is very important in F.T. theory.



The Dirac delta function

The Dirac delta function, or impulse function, $\delta(t)$, is another important function in F.T. theory.

It is defined as follows:

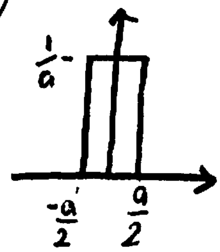
$$\int_{-\infty}^{\infty} \delta(t-t_0) . x(t) \, dt = x(t_0) \qquad \circledast$$

where $x(t)$ is an arbitrary function continuous at t_0.

$\delta(t)$ has no real functional meaning if it appears outside an integral. Considering how it is defined, it would need to be zero for all t other than t_0, and would need to have unit area under its "curve", which means that it must be an infinitely high, zero width "spike" of area one! This is not a function in the mathematical sense but, using its definition $\circledast$, it proves immensely useful.

You will sometimes see $\delta(t)$ defined as follows:

Take a function $r_a(t) = \begin{cases} \frac{1}{a}, & |t| < \frac{a}{2} \\ 0, & \text{otherwise} \end{cases}$



then $\delta(t) = \lim\limits_{a \to 0} r_a(t)$

Another form which proves important is:

$$\delta(t) = \lim_{a \to \infty} \frac{\sin at}{\pi t}$$

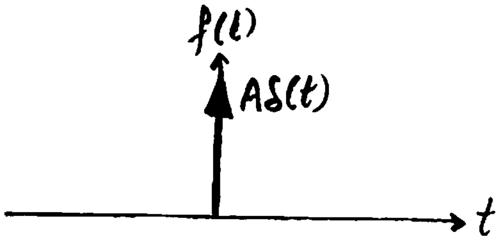
The Fourier transforms of an impulse, a constant, and a sinusoid

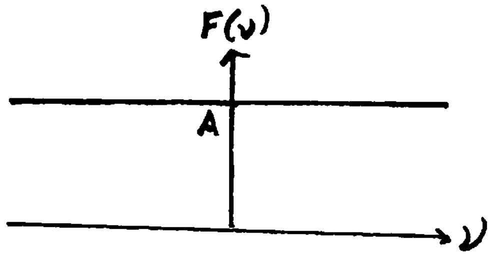
Take $f(t) = A.\delta(t)$

$$F(\nu) = \int_{-\infty}^{\infty} A.\delta(t).\, e^{-i2\pi\nu t}\, dt$$

by the definition of $\delta(t)$ this becomes:

$$F(\nu) = A.e^{-i2\pi\nu.0} = Ae^{0} = A$$





So the F.T. of an impulse at $t=0$ is a constant.

— // —

Take $f(t) = A$

$$F(\nu) = \int_{-\infty}^{\infty} Ae^{-i2\pi\nu t}\, dt$$

$$= \frac{A}{-i2\pi\nu} e^{-i2\pi\nu t}\Big|_{-\infty}^{\infty}$$

$$= \lim_{a\to\infty} \frac{A}{-i2\pi\nu}\left(e^{-i2\pi\nu a} - e^{i2\pi\nu a}\right)$$

$$= \lim_{a\to\infty} \frac{A}{\pi\nu} \sin 2\pi\nu a$$

$$= A.\delta(\nu)$$

(by the limit at the bottom of the previous page)

So the F.T. of a constant is an impulse at $\nu=0$

Now consider a sinusoid

$$f(t) = A\cos 2\pi\nu_0 t$$

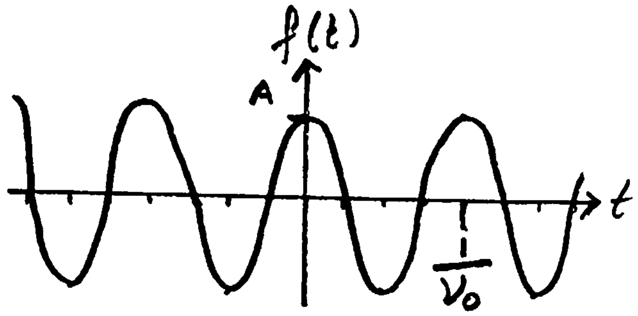
$$= \frac{A}{2}\left(e^{i2\pi\nu_0 t} + e^{-i2\pi\nu_0 t}\right)$$

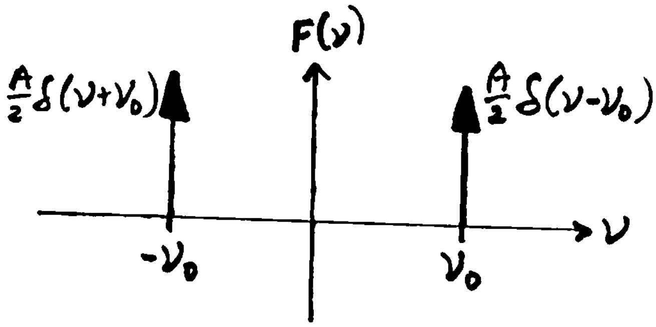
$$F(\nu) = \int_{-\infty}^{\infty} f(t)\, e^{-i2\pi\nu t}\, dt$$

$$= \frac{A}{2}\int_{-\infty}^{\infty} e^{-i2\pi t(\nu-\nu_0)} + e^{-i2\pi t(\nu+\nu_0)}\, dt$$

$$= \frac{A}{2}\delta(\nu-\nu_0) + \frac{A}{2}\delta(\nu+\nu_0)$$

So the F.T. of a cosine of amplitude A, is two delta functions, with weighting of $A/2$, one at $\nu = \nu_0$ and one at $\nu = -\nu_0$





Compare these continuous Fourier transforms against the equivalent Fourier series. Where the Fourier series would have had a coefficient of magnitude $\frac{A}{2}$, the Fourier transform has a delta function weighted by factor $A/2$.

The delta function is vital to an understanding of sampling theory: that is, of how the continuous relates to the discrete.

Properties of the Fourier transform

Linearity:

$$\text{if } f(t) \overset{F.T.}{\Longleftrightarrow} F(\nu)$$
$$\text{and } g(t) \Longleftrightarrow G(\nu)$$
$$\text{then } f(t) + g(t) \Longleftrightarrow F(\nu) + G(\nu)$$
$$\text{and } k.f(t) \Longleftrightarrow k.F(\nu)$$

Symmetry:

$$\text{if } f(t) \overset{F.T.}{\Longleftrightarrow} F(\nu)$$
$$\text{then } F(t) \Longleftrightarrow f(-\nu)$$

{ you may need a constant scaling factor to make this work

Scaling:

$$\text{if } f(t) \overset{F.T.}{\Longleftrightarrow} F(\nu)$$
$$\text{then } f(a.t) \Longleftrightarrow \frac{1}{|a|} F\left(\frac{\nu}{a}\right)$$
$$\text{and } \frac{1}{|b|} f\left(\frac{t}{b}\right) \Longleftrightarrow F(b.\nu)$$

Shifting:

$$\text{if } f(t) \overset{F.T.}{\Longleftrightarrow} F(\nu)$$
$$\text{then } f(t - t_0) \Longleftrightarrow e^{-i2\pi\nu t_0} F(\nu)$$
$$\text{and } f(t) e^{i2\pi\nu_0 t} \Longleftrightarrow F(\nu - \nu_0)$$

Even & odd: if $f(t)$ is <u>even</u>, $F(\nu)$ is purely real
if $f(t)$ is <u>odd</u>, $F(\nu)$ is purely imaginery

any $f(t)$ can be turned into the sum of even and odd parts:

$$f(t) = \frac{f(t)}{2} + \frac{f(t)}{2} = \underbrace{\left[\frac{f(t)}{2} + \frac{f(-t)}{2}\right]}_{\text{EVEN}} + \underbrace{\left[\frac{f(t)}{2} - \frac{f(-t)}{2}\right]}_{\text{ODD}}$$

Differentiation: if $f(t) \overset{F.T.}{\Longleftrightarrow} F(\nu)$

then $\frac{d^m f(t)}{dt^m} \Longleftrightarrow (i 2\pi\nu)^m F(\nu)$

this is a useful little theorem: taking derivatives in time is equivalent to a simple multiplication in frequency

Convolution

Convolution is, probably, the most important property in Fourier transform theory. The convolution of two functions $f(t)$ and $g(t)$ is:

$$y(t) = f(t) * g(t) = \int_{-\infty}^{\infty} f(\tau) g(t-\tau)\, d\tau$$

The attached extract from Brigham's book is the best explanation of this integral that I have seen.

The importance of convolution lies in the relationship that:

"convolution in time means multiplication in frequency"

so if $f(t) \Longleftrightarrow F(\nu)$
and $g(t) \Longleftrightarrow G(\nu)$
then $f(t) * g(t) \Longleftrightarrow F(\nu).G(\nu)$

the converse is also true:

$$f(t).g(t) \Longleftrightarrow F(\nu) * G(\nu)$$

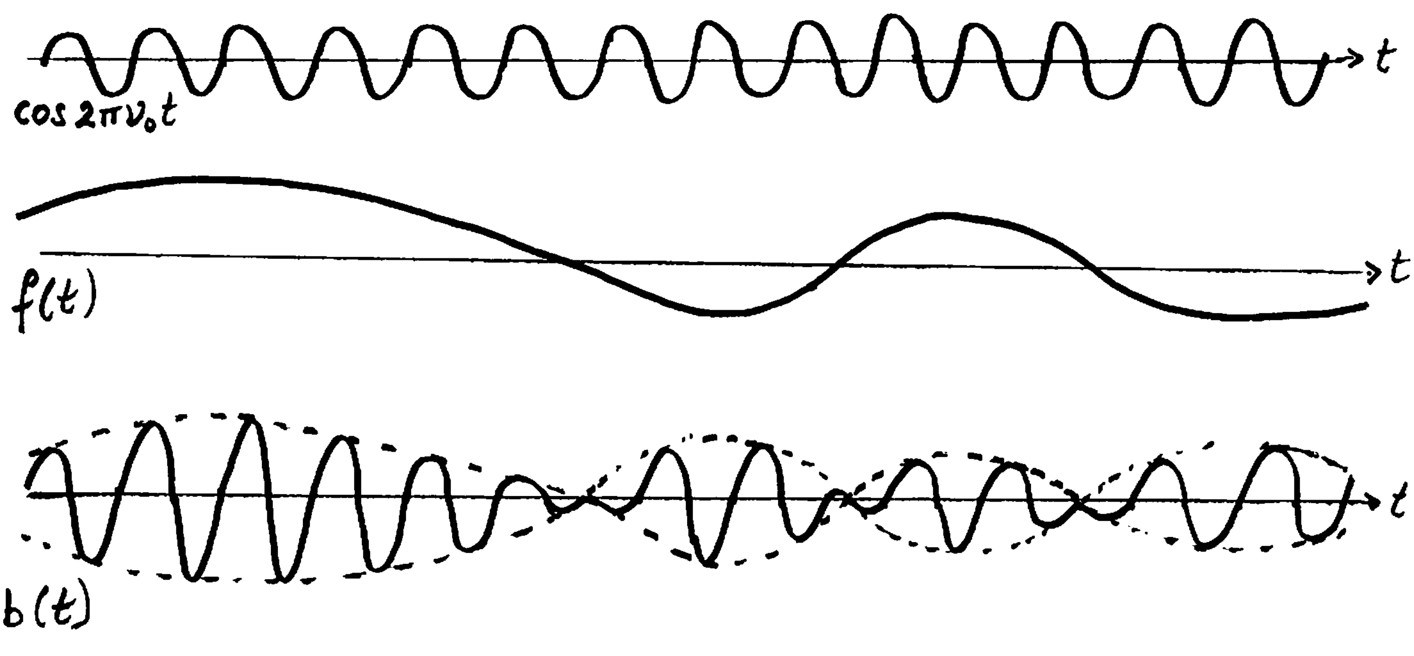
Convolution Examples

The extract from Brigham's book contains a couple of simple examples.

A further example is amplitude-modulated (AM) radio. If we take a sound wave, $f(t)$, which will cover the range 0–16 kHz (the range of frequencies which a human can hear) we can broadcast it by modulating it on a radio frequency carrier wave. In other words, the broadcast signal, $b(t)$, is $f(t)$ multiplied by a sinusoid:

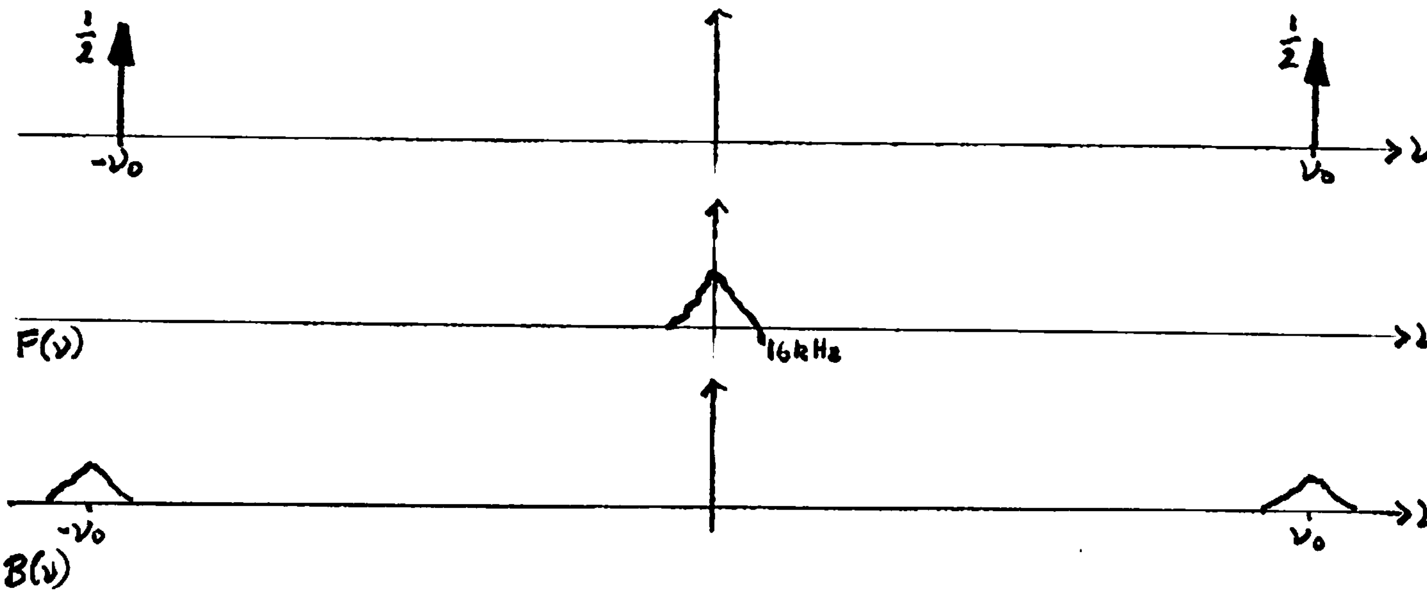
$$b(t) = f(t) . \cos(2\pi \nu_0 t)$$

In the UK ν_0 will be in the range 144–1611 kHz for commercial radio. Note that ν_0 is a much higher frequency than even the highest frequency in $f(t)$.



What happens in the frequency domain? Well, the F.T. of $f(t)$ is CONVOLVED with the F.T. of the sinusoid.

Thus: $$\mathcal{B}(\nu) = F(\nu) * \left[\tfrac{1}{2}\delta(\nu-\nu_0) + \tfrac{1}{2}\delta(\nu+\nu_0)\right]$$
$$= \tfrac{1}{2}F(\nu-\nu_0) + \tfrac{1}{2}F(\nu+\nu_0)$$



How do I now recover the original signal? It's simple: I simply multiply $b(t)$ by a sinusoid of the same frequency: $$\hat{f}(t) = b(t).\cos(2\pi\nu_0 t)$$

In the frequency domain we can visualise what this means:
$$\hat{F}(\nu) = \mathcal{B}(\nu) * \left[\tfrac{1}{2}\delta(\nu-\nu_0) + \tfrac{1}{2}\delta(\nu+\nu_0)\right]$$
$$= \tfrac{1}{4}F(\nu-2\nu_0) + \tfrac{1}{2}F(\nu) + \tfrac{1}{4}F(\nu+2\nu_0)$$

This is thus the original signal (scaled by $\frac{1}{2}$) with an extraneous copy centred at $\nu = 2\nu_0$.

Now, $\hat{f}(t)$ will be played through a loudspeaker. No loudspeaker can handle frequencies up around $2\nu_0$, so the loudspeaker, which is of course a linear system, gets rid of that extraneous copy, leaving just the original signal.

Furthermore, even if the loudspeaker could produce sound waves with frequencies around $2\nu_0$, your ear cannot detect them, again leaving just the original signal.

[Now, it is very easy to make an electrical circuit that outputs a sinusoid at radio frequencies and so the electronics required to make a radio receiver (for AM signals) are very simple indeed. You will, however, have to stick a decent amplifier between the receiver and the loudspeaker]

Now, AM signals in the UK are broadcast in bands that are only 9 kHz apart — which means that you can broadcast sounds up to (at best) $4\frac{1}{2}$ kHz.

So: AM radio does not handle all the high frequency overtones which colour real-world sound. FM radio (with 50kHz wide bands and frequency modulation) does handle them. It also means that an AM receiver needs electronics to filter out all frequencies over $4\frac{1}{2}$ kHz once it has de-modulated the radio signal.

For a demonstration of the aural difference between AM and FM flip your radio between LW 198kHz and FM 93·30MHz when both are broadcasting the same programme.

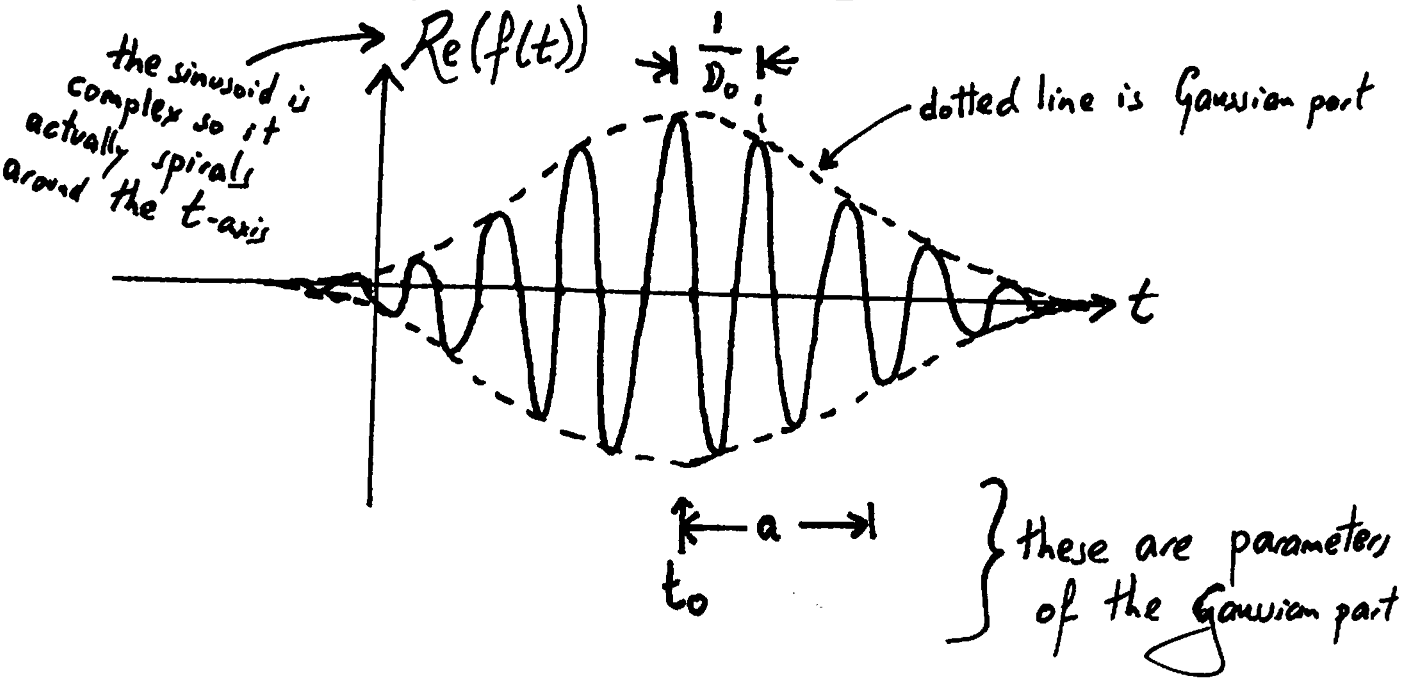
Gabor wavelets – an example of the F.T. in action

A Gabor wavelet is a sinusoid (well, actually it really is a complex exponential) multiplied by a Gaussian. So it represents a signal of a particular frequency (the complex sinusoid bit) localised in a particular part of the domain (that's the Gaussian bit).

Its functional form is:

$$f(t) = e^{-i2\pi\nu_0(t-t_0)} e^{-(t-t_0)^2/a^2}$$

So: the sinusoid has frequency ν_0
the wavelet is localised around t_0
and the spread is determined by a



Gabor wavelets have some interesting properties and prove useful in Computer Vision applications. You can expect more detail in the Computer Vision and Information Theory & Coding courses.

We can find the F.T. of this Gabor wavelet using our knowledge of the F.T. of the Gaussian [Fig 2.12 from Brigham], the shifting property of the F.T. (page F17) and the F.T. of the complex exponential:

$$e^{-i2\pi\nu_0 t} \overset{F.T.}{\Leftrightarrow} \delta(\nu-\nu_0)$$

You should use these to prove that:

$$F(\nu) = e^{-i2\pi\nu t_0} e^{-\pi^2(\nu-\nu_0)^2 a^2} . (\sqrt{\pi} . a)$$

Which is really just $f(t)$ with t and ν, t_0 and ν_0, a and $\frac{1}{a}$ interchanged. Close inspection will show that I have not quite produced a perfect copy of $f(t)$ with the variables simply interchanged. (look at the complex exponentials term...)

If, however, we select a family of Gabor wavelets defined by $(\nu_i, t_{i,j}, a_{i,j,k})$ such that:

$$e^{-i2\pi\nu_i(t-t_{ij})} = e^{-i2\pi\nu_i t} \quad \forall i,j$$

then (apart from the constant scale factor $\sqrt{\pi}.a_{i,j,k}$) the two forms are exactly the same:

$$f(t) = e^{-i2\pi\nu_0 t} e^{-(t-t_0)^2/a^2}$$

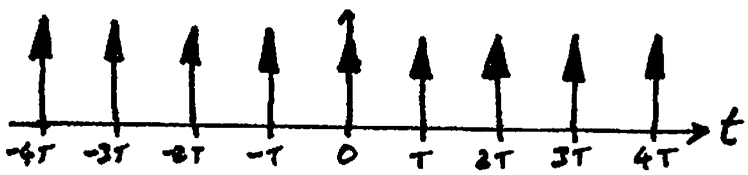
$$F(\nu) = (\sqrt{\pi}a) e^{-i2\pi t_0 \nu} e^{-(\nu-\nu_0)^2 \pi^2 a^2}$$

This family is a "self-Fourier" set of functions, which means that the F.T. of one member of the family is another member of the same family.

The Comb function & sampling

A comb is defined as a sum of infinitely many equispaced Dirac delta functions:

$$\mathrm{comb}(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$



The F.T. of a comb is another comb, with teeth $1/T$ apart:

$$\mathrm{Comb}(\nu) = \sum_{n=-\infty}^{\infty} \frac{1}{T} \delta\left(\nu - \frac{n}{T}\right)$$

//

Sampling is generally thought of as a conversion from a continuous to a discrete function.

e.g. for samples at separation T we can say:

$$f_n = f(n.T)$$

However, f_n is a set of discrete values. We cannot perform a continuous Fourier transform on it. So we define the continuous version of sampling to be:

$$\hat{f}(t) = f(t).\,\mathrm{comb}(t)$$
$$= \sum_{n=-\infty}^{\infty} f(nT).\,\delta(t - nT)$$

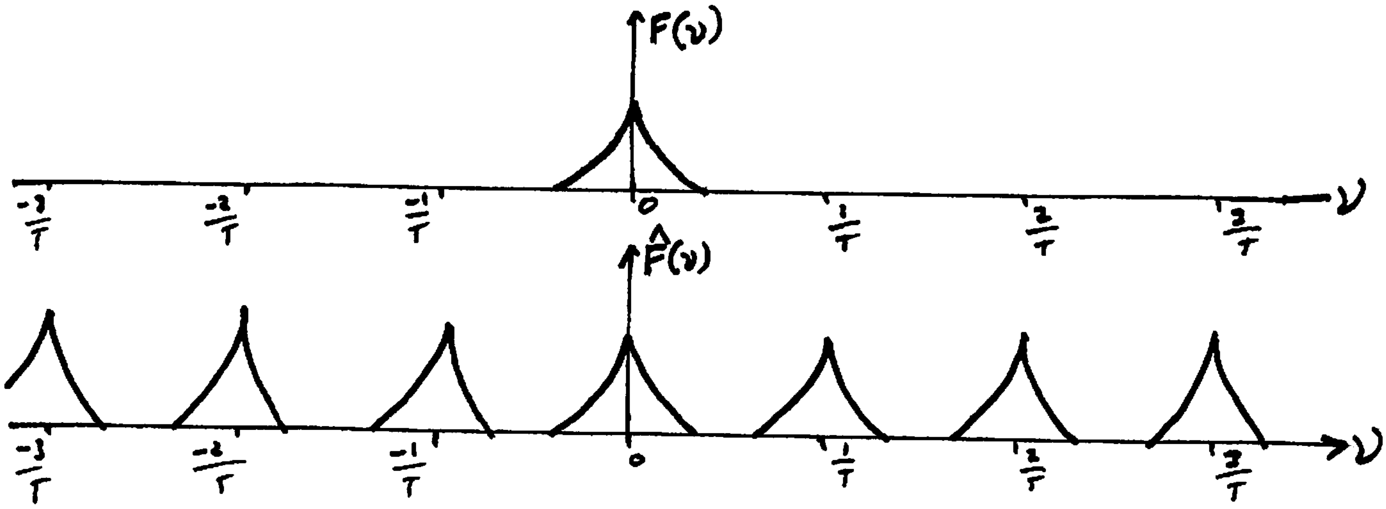
Thus the continuous form of the samples is a set of equispaced, weighted delta functions with the weights being the values of the function, $f(t)$ at the sample point.

The Fourier transform of $\hat{f}(t)$ is:

$$\hat{F}(\nu) = F(\nu) * \text{Comb}(\nu)$$

Convolving a function $F(\nu)$ with a delta function $\delta(\nu - \frac{n}{T})$ creates a shifted copy: $F(\nu - \frac{n}{T})$. So convolving $F(\nu)$ by a Comb produces a whole set of equispaced copies:

$$\hat{F}(\nu) = \sum_{n=-\infty}^{\infty} F\left(\nu - \frac{n}{T}\right)$$



So, how do we recover the original?

Simply multiply $\hat{F}(\nu)$ by a box function

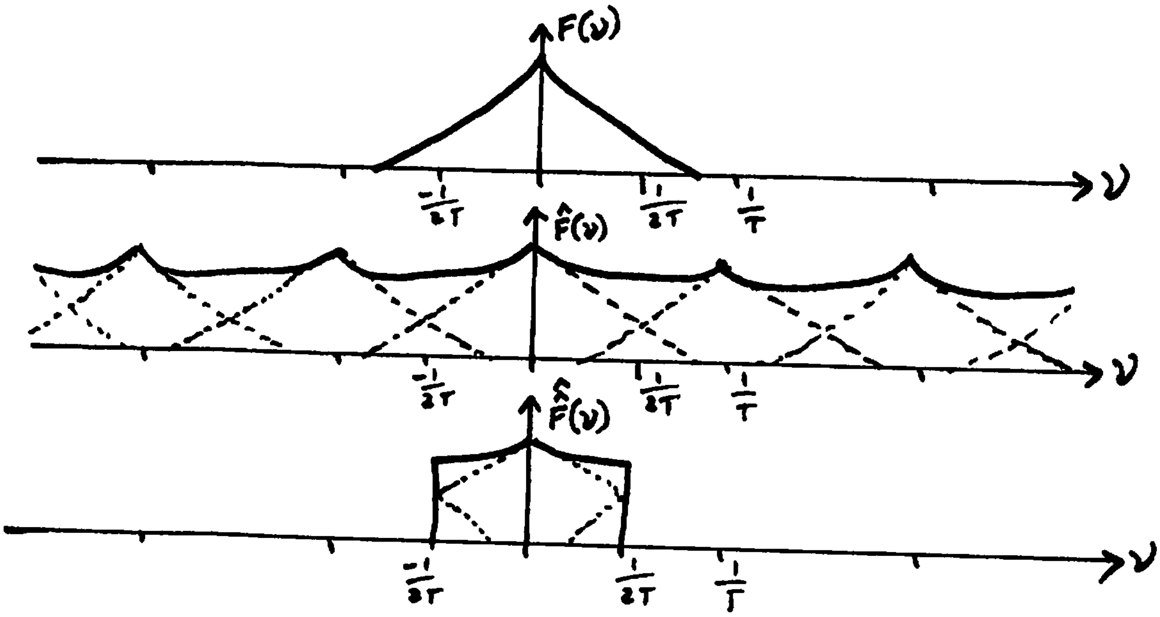
$$\hat{\hat{F}}(\nu) = \hat{F}(\nu) . \text{Box}(\nu)$$

$$Box(\nu) = \begin{cases} 1, & |\nu| < \frac{1}{2T} \\ \frac{1}{2}, & |\nu| = \frac{1}{2T} \\ 0, & |\nu| > \frac{1}{2T} \end{cases}$$

This multiplication gets rid of all the spurious copies and restores the original function PROVIDED

$$F(\nu) = 0, \quad |\nu| \geq \frac{1}{2T}$$

If this isn't true then the copies overlap and sum up, so the Box restores something where all frequencies greater than $\frac{1}{2\pi}$ map to (or alias to) frequencies less than $\frac{1}{2\pi}$, hence $\hat{\hat{F}}(\nu) \neq F(\nu)$ unless the above condition holds.



Nyquist's sampling theorem states that, if:

$$F(\nu) = 0, \quad |\nu| \geq \frac{1}{2T}$$

then $f(t)$ is completely described by the samples f_n, taken at separation T.

The above analysis shows why this is so.

So, if we wish to reconstruct $f(t)$ from $\hat{f}(t)$, how do we do it?

We know that the F.T. of Box (ν) is:

$$\frac{\sin \frac{2\pi}{T} t}{2\pi t}$$

and we know that convolution in one domain implies multiplication in the other, and vice-versa, so:

$$\hat{\hat{f}}(t) = \hat{f}(t) * \left(\frac{\sin \frac{2\pi}{T} t}{2\pi t}\right)$$

$$= \sum_{n=-\infty}^{\infty} f(nT) . \frac{\sin\left[\frac{2\pi}{T}(t-nT)\right]}{2\pi . (t-nT)}$$

$$\left[= f(t) \quad \text{IFF} \quad F(\nu) = 0, |\nu| \geq \frac{1}{2T} \right]$$

So, the reconstructed function is the sum of weighted sinc functions. Whacky but true!

If $f(t) \neq 0 \quad \forall \nu, |\nu| \geq \frac{1}{2T}$ then we get artefacts in the reconstructed signal. We call these artefacts "aliasing".

If we use some function other than sinc to try to recover the original signal we get a whole different set of artefacts.

The Fourier transform and the Fourier series

To explain the Fourier series in terms of the Fourier transform we can follow this reasoning:

A Fourier series represents a function, $f(t)$, which is periodic with period T.

$$f(t) = f(t+T), \quad \forall t$$

Let $f_L(t)$ be one cycle of $f(t)$:

$$f_L(t) = \begin{cases} f(t), & -\frac{T}{2} \leq t < \frac{T}{2} \\ 0, & \text{otherwise} \end{cases}$$

We can now contruct $f(t)$ from $f_L(t)$ using convolution by a comb:

$$f(t) = f_L(t) * \text{comb}(t)$$

If we convolve with a comb in time, we multiply be a Comb in frequency:

$$F(\nu) = F_L(\nu).\,\text{Comb}(\nu)$$

$$= \sum_{k=-\infty}^{\infty} F_L\left(\frac{k}{T}\right).\,\delta\left(\nu - \frac{k}{T}\right)$$

And so, we see that our Fourier series coefficients, c_k, are just the weighting factors on the delta functions:

$$c_k = F_L\left(\frac{k}{T}\right)$$

Finally, we can combine everything together to discover what happens when, in real life, we take a finite length set of N equispaced samples. This can be treated as a periodic function of infinite length:

$$f_n = f_{n+N}$$

and can be treated as a continuous function by the method at the foot of F24.

The Fourier transform is also a sum of weighted Dirac delta functions, and is also periodic with period N:

$$c_k = c_{k+N}$$

We can, thus, drop the delta function and write a computer program to Fourier transform the N sample values $(f_0, \ldots, f_{N-1})$ to the N coefficients $(c_0, \ldots, c_{N-1})$.

This is the Discrete Fourier Transform, which is implemented efficiently as the Fast Fourier Transform (FFT), which is certainly not a subject for a <u>Continuous</u> Mathematics course.

NAD 20/10/00