Partially ordered sets

A binary relation $\sqsubseteq$ on a set D is a partial order iff it is

- reflexive: $d \sqsubseteq d$
- transitive: $d \sqsubseteq d' \sqsubseteq d'' \supset d \sqsubseteq d''$
- anti-symmetric: $d \sqsubseteq d' \sqsubseteq d \supset d = d'$.

Such a pair $(D, \sqsubseteq)$ is called a partially ordered set, or poset.

Cpo's and domains

A(n ω-)chain complete poset, or (ω-)cpo for short, is a poset $(D, \sqsubseteq)$ in which all countable increasing chains $d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \ldots$ have least upper bounds, $\bigsqcup_{n \geq 0} d_n$:

$$\forall m \geq 0 \,.\, d_m \sqsubseteq \bigsqcup_{n \geq 0} d_n$$

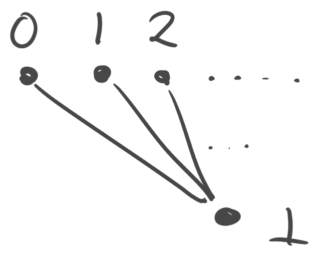
$$\forall d \in D \,.\, (\forall m \geq 0 \,.\, d_m \sqsubseteq d) \ \supset \ \bigsqcup_{n \geq 0} d_n \sqsubseteq d$$

A domain is a cpo that possesses a least element, $\bot$:

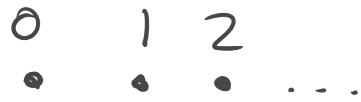
$$\forall d \in D \,.\, \bot \sqsubseteq d$$

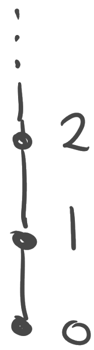
Domains

Examples



Non-examples





Partial functions

The set $X \rightharpoonup Y$ of partial functions from a set X to a set Y is a domain with

- Partial order: $f \sqsubseteq g$ iff $dom(f) \subseteq dom(g)$ and $\forall x \in dom(f).\ f(x) = g(x)$.
- Lub of chain $f_0 \sqsubseteq f_1 \sqsubseteq f_2 \sqsubseteq \ldots$ is the partial function f with $dom(f) = \bigcup_{n \geq 0} dom(f_n)$ and

$$f(x) = \begin{cases} f_n(x) & \text{if } x \in dom(f_n) \text{ for some } n \\ \text{undefined} & \text{otherwise} \end{cases}$$

- Least element $\perp$ = totally undefined partial function.

Monotonicity, continuity, strictness

- A function $f : D \to E$ between posets is monotone iff $\forall d, d' \in D.\ d \sqsubseteq d' \supset f(d) \sqsubseteq f(d')$.
- If D and E are cpos, the function f is continuous iff it is monotone and preserves lubs of chains, i.e. for all chains $d_0 \sqsubseteq d_1 \sqsubseteq \dots$ in D, it is the case that

$$f(\bigsqcup_{n \geq 0} d_n) = \bigsqcup_{n \geq 0} f(d_n) \quad \text{in } E$$

- If D and E have least elements, then the function f is strict iff $f(\bot) = \bot$.

Least pre-fixed points

Let D be a poset and $f : D \to D$ be a function.

An element $d \in D$ is a pre-fixed point of f if it satisfies $f(d) \sqsubseteq d$.

The least pre-fixed point of f, if it exists, will be written $fix(f)$. It is thus (uniquely) specified by the two properties:

$$f(fix(f)) \sqsubseteq fix(f)$$
$$\forall d \in D.\ f(d) \sqsubseteq d \ \supset\ fix(f) \sqsubseteq d$$

These imply that $fix(f)$ is a fixed point of f, that is,

$$f(fix(f)) = fix(f)$$

Tarski's Fixed Point Theorem

Let $f : D \to D$ be a continuous function on a domain D. Then f possesses a least pre-fixed point, given by

$$\mathit{fix}(f) = \bigsqcup_{n \geq 0} f^n(\bot)$$

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Proof. By continuity of f,
$f(\bigsqcup_{n \geq 0} f^n(\bot)) = \bigsqcup_{n \geq 0} f(f^n(\bot)) = \bigsqcup_{n \geq 0} f^{n+1}(\bot) = \bigsqcup_{n \geq 1} f^n(\bot) = \bigsqcup_{n \geq 0} f^n(\bot)$; and if $f(d) \sqsubseteq d$, then

- $f^0(\bot) = \bot \sqsubseteq d$
- $f^n(\bot) \sqsubseteq d$ implies $f^{n+1}(\bot) = f(f^n(\bot)) \sqsubseteq f(d) \sqsubseteq d$

so $\bigsqcup_{n \geq 0} f^n(\bot) \sqsubseteq d$.

Plotkin's Uniformity Principle

Suppose μ is an operation assigning to each domain D and continuous function $f : D \to D$ an element $\mu_D(f) \in D$. Then $\mu = \mathit{fix}$ if and only if μ satisfies properties (F) and (U).

(F) $f(\mu_D(f)) = \mu_D(f)$

(U) If
$$\begin{array}{ccc} D & \xrightarrow{s} & D' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ D & \xrightarrow[s]{} & D' \end{array}$$
commutes (i.e. $f' \circ s = s \circ f$)

with f, f', s continuous and s strict,

then $s(\mu_D(f)) = \mu_{D'}(f')$.

$\mu = \mathit{fix} \supset \mu$ satisfies (F) & (U)

(F) — least pre-fixed points are fixed points.

(U):

$$\begin{aligned} s(\mathit{fix}(f)) &= s\left(\bigsqcup_{n \geq 0} f^n(\bot)\right) \\ &= \bigsqcup_{n \geq 0} s(f^n(\bot)) && \text{since } s \text{ continuous} \\ &= \bigsqcup_{n \geq 0} (f')^n(s(\bot)) && \text{since } s \circ f = f' \circ s \\ &= \bigsqcup_{n \geq 0} (f')^n(\bot) && \text{since } s \text{ strict} \\ &= \mathit{fix}(f') \end{aligned}$$

μ satisfies (F) & (U) $\supset$ $\mu = fix$

Let Ω be the domain $\{0 \sqsubseteq 1 \sqsubseteq 2 \sqsubseteq \cdots \sqsubseteq \omega\}$ and $s : \Omega \to \Omega$ the continuous function

$$\begin{cases} s(n) = n+1 \\ s(\omega) = \omega \end{cases}$$

__NB__ ω is the __unique__ fixed point of s, so by (F) we must have $\mu_\Omega(s) = \omega$.

μ satisfies (F) & (U) $\supset$ $\mu = fix$

Given any continuous $f : D \to D$, define a strict continuous function $\hat{f} : \Omega \to D$ by

$$\begin{cases} \hat{f}(n) = f^n(\bot) \\ \hat{f}(\omega) = fix(f). \end{cases}$$

Thus
$$\begin{array}{ccc} \Omega & \xrightarrow{\hat{f}} & D \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle f} \\ \Omega & \xrightarrow[\hat{f}]{} & D \end{array}$$
commutes, so by (U) we have

$$\mu_D(f) = \hat{f}(\mu_\Omega(s)) = \hat{f}(\omega) = fix(f)$$

$\square$