Topics in Logic and Complexity
Handout 8

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Expressive Power of Logics

We have seen that the expressive power of *first-order logic*, in terms of computational complexity is *weak*.

Second-order logic allows us to express all properties in the *polynomial hierarchy*.

Are there interesting logics intermediate between these two?

We have seen one—*monadic second-order logic*.

We now examine another—*LFP*—the logic of *least fixed points*.

Inductive Definitions

LFP is a logic that formalises *inductive definitions*.

> Unlike in second-order logic, we cannot quantify over *arbitrary* relations, but we can build new relations *inductively*.

Inductive definitions are pervasive in mathematics and computer science.

The *syntax* and *semantics* of various formal languages are typically defined inductively.

> *viz.* the definitions of the syntax and semantics of first-order logic seen earlier.

Transitive Closure

The *transitive closure* of a binary relation E is the *smallest* relation T satisfying:

- $E \subseteq T$; and
- if $(x, y) \in T$ and $(y, z) \in E$ then $(x, z) \in T$.

This constitutes an *inductive definition* of T and, as we have already seen, there is no *first-order* formula that can define T in terms of E.

Monotone Operators

In order to introduce LFP, we briefly look at the theory of *monotone operators*, in our restricted context.

We write $\mathsf{Pow}(A)$ for the powerset of A.

An operator in A is a function

$$F : \mathsf{Pow}(A) \to \mathsf{Pow}(A).$$

F is *monotone* if

$$\text{if } S \subseteq T, \text{ then } F(S) \subseteq F(T).$$

Least and Greatest Fixed Points

A *fixed point* of F is any set $S \subseteq A$ such that $F(S) = S$.

S is the *least fixed point* of F, if for all fixed points T of F, $S \subseteq T$.

S is the *greatest fixed point* of F, if for all fixed points T of F, $T \subseteq S$.

Least and Greatest Fixed Points

For any monotone operator F, define the collection of its *pre-fixed points* as:

$$Pre = \{S \subseteq A \mid F(S) \subseteq S\}.$$

Note: $A \in Pre$.

Taking

$$L = \bigcap Pre,$$

we can show that L is a fixed point of F.

Fixed Points

For any set $S \in Pre$,

$L \subseteq S$	by definition of L.
$F(L) \subseteq F(S)$	by monotonicity of F.
$F(L) \subseteq S$	by definition of Pre.
$F(L) \subseteq L$	by definition of L.
$F(F(L)) \subseteq F(L)$	by monotonicity of F
$F(L) \in Pre$	by definition of Pre.
$L \subseteq F(L)$	by definition of L.

Least and Greatest Fixed Points

L is a *fixed point* of F.

Every fixed point P of F is in Pre, and therefore $L \subseteq P$.

Thus, L is the least fixed point of F

Similarly, the greatest fixed point is given by:

$$G = \bigcup \{S \subseteq A \mid S \subseteq F(S)\}.$$

Iteration

Let A be a *finite* set and F be a *monotone* operator on A.

Define for $i \in \mathbb{N}$:

$$\begin{aligned} F^0 &= \emptyset \\ F^{i+1} &= F(F^i). \end{aligned}$$

For each i, $F^i \subseteq F^{i+1}$ (proved by induction).

Iteration

Proof by induction.

$$\emptyset = F^0 \subseteq F^1.$$

If $F^i \subseteq F^{i+1}$ then, by monotonicity

$$F(F^i) \subseteq F(F^{i+1})$$

and so $F^{i+1} \subseteq F^{i+2}$.

Fixed-Point by Iteration

If A has n elements, then

$$F^n = F^{n+1} = F^m \quad \text{for all} \quad m > n$$

Thus, F^n is a fixed point of F.

Let P be any fixed point of F. We can show induction on i, that $F^i \subseteq P$.

$$F^0 = \emptyset \subseteq P$$

If $F^i \subseteq P$ then

$$F^{i+1} = F(F^i) \subseteq F(P) = P.$$

Thus F^n is the *least fixed point* of F.

Defined Operators

Suppose ϕ contains a relation symbol R (of arity k) not interpreted in the structure $\mathbb{A}$ and let $\mathbf{x}$ be a tuple of k free variables of ϕ.

For any relation $P \subseteq A^k$, ϕ defines a new relation:

$$F_P = \{\mathbf{a} \mid (\mathbb{A}, P) \models \phi[\mathbf{a}]\}.$$

The operator $F_\phi : \mathsf{Pow}(A^k) \to \mathsf{Pow}(A^k)$ defined by ϕ is given by the map

$$P \mapsto F_P.$$

Or, $F_{\phi,\mathbf{b}}$ if we fix parameters $\mathbf{b}$.

Positive Formulas

Definition
A formula ϕ is *positive* in the relation symbol R, if every occurence of R in ϕ is within the scope of an even number of negation signs.

Lemma
For any structure $\mathbb{A}$ not interpreting the symbol R, any formula ϕ which is positive in R, and any tuple $\mathbf{b}$ of elements of A, the operator $F_{\phi,\mathbf{b}} : \mathsf{Pow}(A^k) \to \mathsf{Pow}(A^k)$ is monotone.

Reading List for this Handout

1. Ebbinghaus and Flum. Section 8.1.
2. Libkin. Sections 10.1 and 10.2.
3. Grädel et al. Section 3.3.