Topics in Logic and Complexity
Handout 12

Anuj Dawar

MPhil Advanced Computer Science, Lent 2010

Types

Recall:
For a tuple $\mathbf{a}$ in $\mathbb{A}$, $\mathsf{Type}^k(\mathbb{A}, \mathbf{a})$ denotes the collection of all formulas $\phi \in L^k$ such that $\mathbb{A} \models \phi[\mathbf{a}]$.

That is, $(\mathbb{A}, \mathbf{a}) \equiv^k (\mathbb{B}, \mathbf{b})$ if, and only if,

$$\mathsf{Type}^k(\mathbb{A}, \mathbf{a}) = \mathsf{Type}^k(\mathbb{B}, \mathbf{b})$$

For every finite structure $\mathbb{A}$, for every $l \leq k$ and l-tuple $\mathbf{a}$ of elements from $\mathbb{A}$, there is a formula, $\phi \in \mathsf{Type}^k(\mathbb{A}, \mathbf{a})$ such that for any structure $\mathbb{B}$ and l-tuple $\mathbf{b}$ of elements of $\mathbb{B}$, $\mathbb{B} \models \phi[\mathbf{b}]$ if, and only if, $(\mathbb{A}, \mathbf{a}) \equiv^k (\mathbb{B}, \mathbf{b})$.

Atomic Types

An *atomic type* is the conjunction of a *maximally consistent* set of atomic and negated atomic formulas, in the variables $x_1, \ldots, x_k$.

Note that, in a *finite* vocabulary, an atomic type is a quantifier-free first-order formula.

For each structure $\mathbb{A}$ and $\mathbf{a} \in A^k$ there is a *unique* atomic type $\tau(x_1, \ldots, x_k)$ such that

$$\mathbb{A} \models \tau[\mathbf{a}].$$

Defining Types

$\mathbf{a} \in A^l$

$\phi^0_{\mathbf{a}}(x_1 \ldots x_l)$ is the conjunction of all atomic and negated atomic formulas $\theta(x_1 \ldots x_l)$ such that $\mathbb{A} \models \theta[\mathbf{a}]$. That is, it is the *atomic type* of $\mathbf{a}$ in $\mathbb{A}$.

$$\begin{aligned}
\phi^{p+1}_{\mathbf{a}} &= \phi^p_{\mathbf{a}} \wedge \bigwedge_{a \in A} \exists x_{l+1} \phi^p_{\mathbf{a}a} \wedge \forall x_{l+1} \bigvee_{a \in A} \phi^p_{\mathbf{a}a} && (\text{if } l < k) \\
\phi^{p+1}_{\mathbf{a}} &= \phi^p_{\mathbf{a}} \wedge \bigwedge_{i=1\ldots k} \phi^{p+1}_{\mathbf{a}_i} && (\text{if } l = k)
\end{aligned}$$

where $\mathbf{a}_i$ is obtained from $\mathbf{a}$ by removing a_i.

Defining Types

$\mathbb{A} \models \phi^p_{\mathbf{a}}[\mathbf{a}']$ *if, and only if,* $(\mathbb{A}, \mathbf{a}) \equiv^k_p (\mathbb{A}, \mathbf{a}')$.

That is, $(\mathbb{A}, \mathbf{a}')$ satisfies the same formulas of L^k of *quantifier rank* at most p as $(\mathbb{A}, \mathbf{a})$.

There is some $q (\leq n^k)$ such that $\equiv^k_{q+1}$ is the same as $\equiv^k_q$ in $\mathbb{A}$.

Take

$$\phi^q_{\mathbf{a}} \wedge \forall \mathbf{x} \big(\bigvee_{\mathbf{a}' \in A^k} \phi^q_{\mathbf{a}'} \wedge \bigwedge_{\mathbf{a}' \in A^k} (\phi^q_{\mathbf{a}'} \leftrightarrow \phi^{q+1}_{\mathbf{a}'}) \big)$$

This formula defines $\mathsf{Type}^k(\mathbb{A}, \mathbf{a})$ among all structures.

Defining Equivalence

There is a formula $\eta(\mathbf{x}, \mathbf{y})$ of IFP which defines $\equiv^k$ in the sense that, for *any* structure $\mathbb{A}$ and tuples $\mathbf{a}, \mathbf{a}' \in A^k$,

$$\mathbb{A} \models \eta[\mathbf{a}, \mathbf{a}'] \quad \text{if, and only if, } (\mathbb{A}, \mathbf{a}) \equiv^k (\mathbb{A}, \mathbf{a}')$$

We construct η by first defining *inductively* the set of positions that are winning for *Spoiler* in the k-pebble game.

Defining Equivalence

Let $\alpha_1(x_1 \ldots x_k), \ldots, \alpha_m(x_1 \ldots x_k)$ be an enumeration, up to equivalence, of all atomic types with k variables on the finite signature σ.

$$\phi_0(\mathbf{xy}) \equiv \bigvee_{1 \leq i \neq j \leq m} (\alpha_i(\mathbf{x}) \wedge \alpha_j(\mathbf{y}))$$

$$\begin{aligned} \phi(R, \mathbf{xy}) \equiv \phi_0(\mathbf{xy}) \quad & \vee \bigvee_{1 \leq i \leq k} \exists x_i \forall y_i R(\mathbf{xy}) \\ & \vee \bigvee_{1 \leq i \leq k} \exists y_i \forall x_i R(\mathbf{xy}) \end{aligned}$$

$$\eta \equiv \neg [\mathbf{ifp}_{R, \mathbf{x}, \mathbf{y}} \phi](\mathbf{x}, \mathbf{y})$$

Ordering the Types

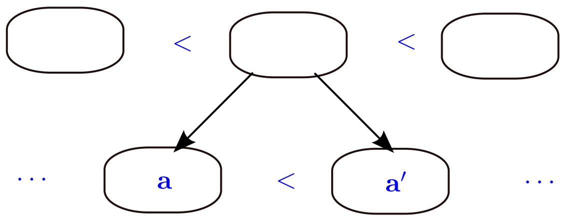
There is an IFP formula, ψ, that defines, in any structure $\mathbb{A}$, an *order* on the equivalence classes of $\equiv^k$, in the sense that,

- on any structure, $\mathbb{A}$, ψ defines a linear pre-order on k-tuples; and
- if $\mathbf{a}$ and $\mathbf{a}'$ have the same type, then neither $\mathbb{A} \models \psi[\mathbf{a}, \mathbf{b}]$ nor $\mathbb{A} \models \psi[\mathbf{b}, \mathbf{a}]$.

The order is defined *inductively*.

To start, choose an arbitrary order on the atomic types $\alpha_1(x_1 \ldots x_k), \ldots, \alpha_m(x_1 \ldots x_k)$.

Ordering the Types



Suppose $\mathbf{a}$ and $\mathbf{a}'$ are equivalent at stage q but in distinct classes at stage $q+1$. Then, for some i, the collection of $\equiv_q^k$ classes that one can get from $\mathbf{a}$ by replacing the ith element is different from the ones you can get from $\mathbf{a}'$.

If the *smallest* (in the ordering so far) class in the symmetric difference is obtainable from $\mathbf{a}$, we put $\mathbf{a}$ before $\mathbf{a}'$ otherwise we put $\mathbf{a}'$ before $\mathbf{a}$.

Interpreting an Ordered Structure

Assume that $\mathbb{A}$ is a structure in a vocabulary σ in which every relation symbol has arity *at most* k.

We associate with $\mathbb{A}$ a structure $I_k(\mathbb{A})$ we call the *k-invariant* of $\mathbb{A}$, in a vocabulary ρ which contains:

- a *binary* relation $<_k$;
- a *unary* relation $='$;
- for each R in σ a *unary* relation R';
- for each i with $1 \leq i \leq k$ a *binary* relation X_i; and
- for each permutation π of the set $\{1, \ldots, k\}$ a *binary* relation P_π.

Interpreting an Ordered Structure

$$I_k(\mathbb{A}) = (A^k/\equiv^k, <_k, =', R'_j, X_i, P_\pi)$$

- Universe $A^k/\equiv^k$
- $<_k$—ordering as defined
- $='$ $([\mathbf{a}])$ iff $\mathbf{a} = (a_1, a_2, \ldots, a_k)$ and $a_1 = a_2$
- $R'([\mathbf{a}])$ iff $\mathbf{a}|_{arity(R)} \in R$
- $X_i([\mathbf{a}], [\mathbf{b}])$ iff $\mathbf{a}$ and $\mathbf{b}$ differ *at most* on their ith element
- $P_\pi([\mathbf{a}], [\mathbf{b}])$ iff $\pi(\mathbf{a}) = \mathbf{b}$.

IFP vs. PFP

Any first order formula ϕ on $I_k(\mathbb{A})$ translates to an IFP formula ϕ^* on $\mathbb{A}$.

That is, $\mathbb{A} \models \phi^*[\mathbf{a}]$ *if, and only if* $I_k(\mathbb{A}) \models \phi[[\mathbf{a}]_{\equiv^k}]$.

ϕ^* is obtained by replacing each relation symbol in ϕ by the IFP formula *defining* it. This includes replacing *equality* by the definition of $\equiv^k$.

IFP vs. PFP

By the same argument, we also have:

- Any IFP formula on $I_k(\mathbb{A})$ translates to a corresponding IFP formula on $\mathbb{A}$.
- Any PFP formula on $I_k(\mathbb{A})$ translates to a corresponding PFP fomula on $\mathbb{A}$.

IFP vs. PFP

Any L^k formula ϕ on $\mathbb{A}$ translates to a corresponding first order fomula ϕ' on $I_k(\mathbb{A})$.

- $R(\mathbf{x})$ gives $R'(x)$. If $\mathbf{y}$ is a permutation of $\mathbf{x}$ such that $\pi(\mathbf{x}) = \mathbf{y}$ then $R(\mathbf{y})$ gives $\exists y(P_\pi(x,y) \wedge R'(y))$.
- $x_i = x_j$ gives $\exists y(P_\pi(x,y) \wedge =' (y))$ where π is a permutation such that $\pi(i) = 1$ and $\pi(j) = 2$.
- $\exists x_i \psi$ gives $\exists y(X_i(x,y) \wedge \psi'(y))$.

IFP vs. PFP

Again, similar arguments show that for each formula of IFP or PFP, there is a k such that the formula can be translated into an IFP or PFP formula respectively, on $I_k(\mathbb{A})$.

Theorem **(Abiteboul, Vianu 1991)**
IFP = PFP if, and only if, P = PSPACE.

Nondeterministic Fixed Points

We can define a *nondeterministic fixed point logic* which bears the same relationship to NP as IFP has to P.
(Abiteboul, Vardi and Vianu)

Given two formulas $\phi_0(R), \phi_1(R)$, define NF^s for every binary string s:

$NF^\epsilon = \emptyset$

$NF^{s\cdot 0} \equiv F_{\phi_0}(NF^s) \cup NF^s$

$NF^{s\cdot 1} \equiv F_{\phi_1}(NF^s) \cup NF^s$

The nondeterministic fixed point of the pair ϕ_0, ϕ_1 is given by: $\bigcup NF^s$.

Relational Complexity

IFP = NFP *if, and only if,* P = NP.

IFP = PFP *if, and only if,* P = PSPACE.

NFP = PFP *if, and only if,* NP = PSPACE.

Relational Complexity

Indeed, one can *characterise* the expressive power of the fixed-point logics by:

A Boolean query Q is expressible in IFP, NFP or PFP, respectively if, and only if, there is a k such that Q is invariant under $\equiv^k$ and the query

$$Q' = \{I_k(\mathbb{A}) \mid \mathbb{A} \in Q\}$$

is computable in P, NP or PSPACE respectively.

Reading List for this Handout

1. Libkin. Chapter 11
2. Ebbinghaus, Flum Section 8.4